Nash Social Welfare for Fair Division of Bads: Normative and Algorithmic Issues

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Based on three papers with

Anna Bogomolnaia, Hervé Moulin, and Elena Yanovskaya

- "Competitive division of a mixed manna", Econometrica, forthcoming
- "Dividing goods *and* bads under additive utilities" arXiv:1610.03745 [cs.GT]
- Dividing goods *or* bads under additive utilities arXiv:1608.01540 [cs.GT]

Motivation

Fair Division without monetary transfers:

how to allocate resources among agents with different preferences in a fair and efficient way?

- **Examples:** division of a common property (partners dissolving their partnership, divorce, inheritance), seats in overdemanded courses, computational resources, office space

- Most of the results in fair division are about goods
  - Exception: E. Peterson, F. Su. (2002, 2009), E. Segal-Halevi (2017) burnt cake cutting
- But many real problems involve bads
  - e.g., house chores, teaching loads, noxious facilities
- Or goods and bads at the same time

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In this talk

We consider:

- divisible items: bads or mixture of goods and bads (mixed manna)
- **The goal:** to extend the **MaxNashProduct rule**[1] to mixed manna.

We will see:

- structural difference between goods and bads problems
- extension of MaxNashProduct is surprising
- algorithmic and economic open questions

[1] the best rule to allocate goods

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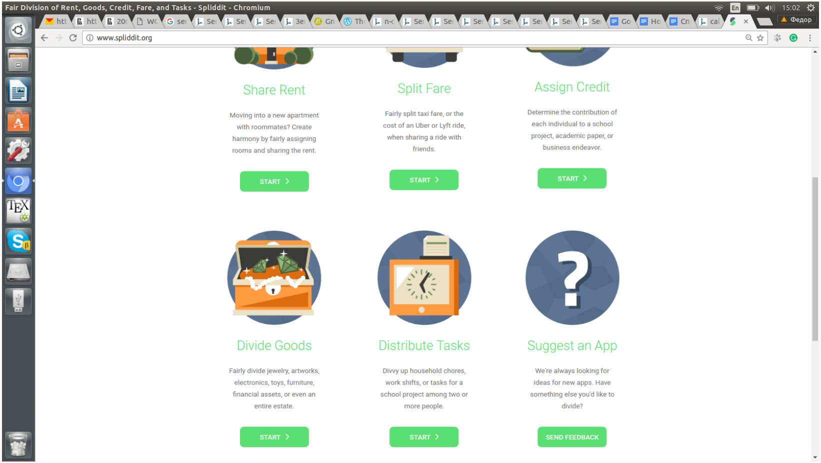
Outline

- Fair division of divisible goods (known results)
    - MaxNashProduct and its properties
    - MaxNashProduct as Competitive Equilibrium for a Fisher market
- Mixture of divisible goods and bads
    - Competitive Equilibrium for mixed manna and extension of MaxNashProduct rule
- All-bads problems with additive utilities
    - Multiplicity issues
    - Algorithms
    - Indivisibilities

Fair division of divisible goods (known results)

How it works on Spliddit.org?

Spliddit.org is launched by the team of Ariel Procaccia (Carnegie Mellon)



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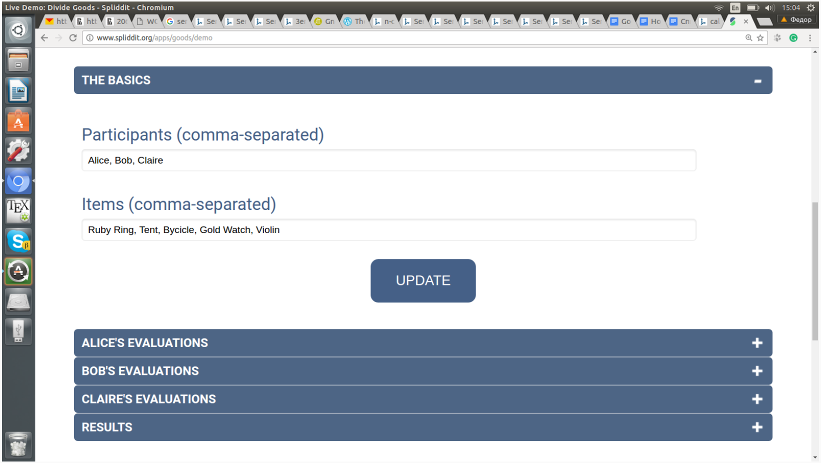

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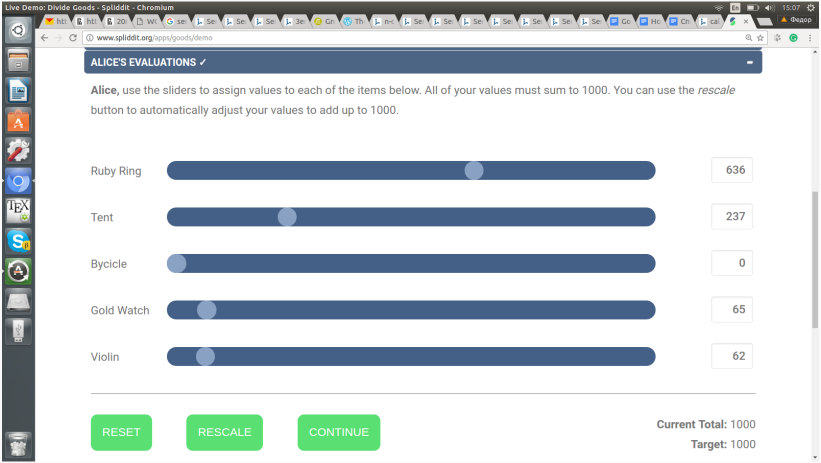

- It is assumed that agents have additive utilities

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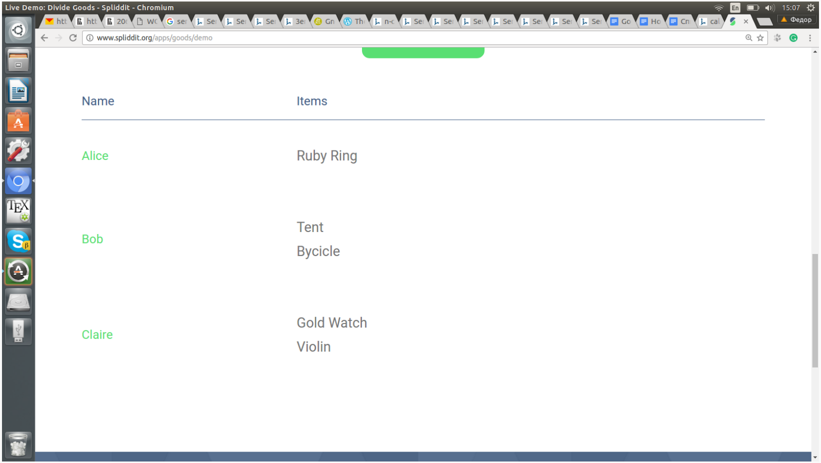

- Spliddit.org uses the MaxNashProduct rule for indivisible items
- Let us look on a simpler **divisible case**

Divisible goods: the model

A fair division problem

- A set of divisible items $M = \{1, 2, ..m\}$, each in the unit amount, is to be distributed among a set of agents $N = \{1, 2, 3.., n\}$
- $z_i = (z_{i1}, z_{i2}, z_{i3}..) \in R_+^M$ is a bundle received by agent i
- an allocation $z = (z_i)_{i \in N}$ is a collection of bundles $z_i \in \mathbb{R}_+^M$ with the condition that all goods are distributed: $\forall a \in M \ \sum_{i \in N} z_{ia} = 1$
- preferences of agent i are given by his utility functions u_i
    - We will focus on **additive utilities**

$$u_i(z_i) = \sum_{a \in M} u_{ia} z_{ia}$$

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$$u_i(z_i) = \sum_{a \in M} u_{ia} z_{ia}$$

Remark: Most of the results remain valid for general monotone, homogeneous, and concave utilities, e.g., Leontief, CES, Cobb-Douglas, etc

Desired properties: Fairness and Efficiency

Envy-Freeness

z is envy-free iff every agent prefers his allocation to the allocation of any other agent:

$$u_i(z_i) \geq u_i(z_j) \text{ for all } i, j \in N.$$

Efficiency

z is efficient iff there is no z' weakly preferred by all agents and by at least one strictly

NashMaxProduct rule

picks an allocation z that maximizes the Nash Social Welfare

$$\mathcal{N}(z) = \prod_{i \in N} u_i(z_i)$$

a similar rule was introduced by J. Nash (1950) in axiomatic bargaining

Properties:

- **Efficient**
- **Envy-Free**
- **Can be efficintly computed**
  - convex problem $\Rightarrow$ approximate solution by gradient methods
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  Proof for $|N| = 2$ with additive utilities:

  Consider an allocation x such that $x_1 = z_1 + \varepsilon z_2$ and $x_2 = (1 - \varepsilon) z_2$. The Nash product can only decrease: $\frac{d}{d\varepsilon} \mathcal{N}(x)|_{\varepsilon=0} \leq 0$. By additivity $\mathcal{N}(x) = (u_1(z_1) + \varepsilon u_1(z_2))(1 - \varepsilon) u_2(z_2)$, and inequality implies $u_1(z_1) \geq u_1(z_2)$. □
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Let us try to guess what could be an extension to problems with bads.

Why goods $\neq$ bads? MaxNashProduct for bads, failed attempts

Bads instead of goods: $u_{ia} \leq 0$ for all agents and items.

Ideas:

- **Minimize the product of disutilities** $\mathcal{N}(z) = \prod_{i \in N} |u_i(z_i)|$
  Very unfair: picks an allocation with $\mathcal{N}(z) = 0$ that gives no bads to one of agents
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To extend MaxNashProduct to bads we will use its connection with **Competitive Equilibrium for a Fisher market**

Back to goods: Fisher Market and its equilibrium

Fisher Market aka Arrow-Debreu exchange economy

- A set M of divisible goods
- A set N of buyers endowed with budgets b_i and utility-functions u_i. Buyers have no value for money.

Allocation z is a **Competitive Equilibrium** if there is a vector $p \in \mathbb{R}_+^M$ of prices such that every agent buys the best bundle he/she can afford, and the market clears. Formally,

$$\forall i \in N : \quad z_i = \mathrm{argmax}_{y \in \mathbb{R}_+^M : \langle y, p \rangle \leq 1} u_i(y).$$

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The Competitive Rule and the MaxNashProduct

The Competitive Rule (CR) (Varian 1974)

aka CEEI, pseudo-market mechanism

Picks a Competitive Equilibrium in a corresponding Fisher Market with equal budgets: $b_i = 1 \;\; \forall i \in N$.

Properties:

- envy-free $\Longleftarrow$ equal choice opportunities
- efficient $\Longleftarrow$ "invisible hand" of Adam Smith

Theorem (Eisenberg (1961), Gale (1960))

Competitive Rule = MaxNashProduct for general homogeneous monotone concave preferences

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Example:

- 4 agents divide 1 hour of a bad "washing the dishes"
- introduce auxiliary good: "not washing"
- 3 hours of "not washing" to distribute, but no agent can consume more than one hour

Corollary: A problem with bads $\Longrightarrow$ a constrained problem with goods.

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Mixture of divisible goods and bads (our results)

The Competitive Rule for mixed manna

Mixture of goods and bads:

- **additive utilities:** u_{ia} of arbitrary sign
- or **concave** ~~**monotone**~~ **homogeneous**

How to define the Competitive Rule?

Allow prices and budgets of both signs.

Basic properties of CR:

- Existence $\Leftarrow$ fixed point arguments from Mas-Colel (1982)
- Envy-Freeness & Efficiency (from standard arguments)

Question: Is it still related to the Nash Social Welfare?

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Relation to the Nash Social Welfare

Main theorem (CR and Nash Social Welfare for mixed manna)

A version of Eisenberg-Gale theorem still holds but now there are three types of problems

- positive, negative, and null

with different behavior of the Competitive Rule.

- The theorem is for general **concave homogeneous** utilities and arbitrary finite sets N and M.
- Illustration: **additive** utilities, 2 **agents and** 3 **items**.

Relation to the Nash Social Welfare

Three items a, b, c, two agents with utilities given by

$$U_1(z_1) = -z_{1a} - 3z_{1b} + \lambda z_{1c}$$

$$U_2(z_2) = -2z_{2a} - z_{2b} + \lambda z_{2c}$$

Parameter $\lambda \geq 0$. Items a, b are bads and c is a good.

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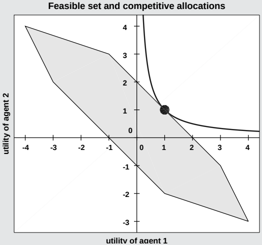
Main theorem (CR and Nash Social Welfare for mixed manna)

- **Positive problems:** the set of feasible utilities intersects positive orthant $(\lambda = 4)$.



CR maximizes the Nash product (similar to all-goods case).

- **Null problems:** knife-edge case. CR picks zero.
- **Negative problems:**

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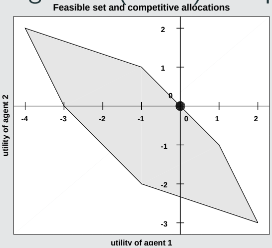

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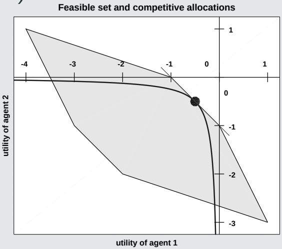

CR picks all critical points of the Nash product on efficient frontier. *Critical point* = local minima, local maxima or sadle-point of Nash Social Welfare on the boundary.

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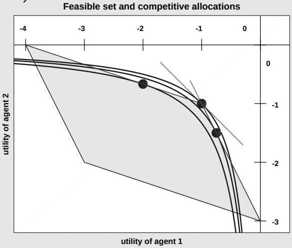

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How to prove? Use an extension of demand-aggregation ideas for homogeneous economies[2].

[2] J. S. Chipman. 1974. Homothetic preferences and aggregation, *Journal of Economic Theory*, 8, 26-38.

Corollary

Analog of MaxNashProduct for all-bads problems

picks all the allocations corresponding to **local minima, local maxima, and saddle points** of the Nash Social Welfare **on the Pareto frontier**

- Envy-Free and Efficient
- Does not solve any convex-optimization problem $\Rightarrow$
  - multiplicity issues
  - algorithmic questions

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All-bads problems with additive utilities

Multiplicity issues & Algorithms & Extension to indivisibilities

Multiplicity issues

Proposition (The number of CR outcomes)

The number of distinct competitive allocations can be as large as $2^{\min\{|M|,|N|\}} - 1$, (exponential growth).

Open question: Any good single-valued selection?

A selection: MaxMinNashProduct rule

1. **Min:** restrict the Nash Social Welfare to the Pareto frontier
2. **Max:** output the allocation that maximizes the restricted product

Open problems: Normative justification? Better selectors?

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Impossibilities

Proposition

For all-bads problems no single-valued rule is:

- Efficient + Envy-Free + Continuous
- Efficient + Fair Share Guaranteed + Resource Monotonic

Remark: in all-goods problems MaxNashProduct satisfies all these axioms. See Megiddo, Vazirani (2007) for Continuity; Segal-Halevi, Sziklai (2015) for Resource Monotonicity.

Corollary:

- All-bads problems are structurally different from the all-goods
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For all-bads problems no single-valued rule is:

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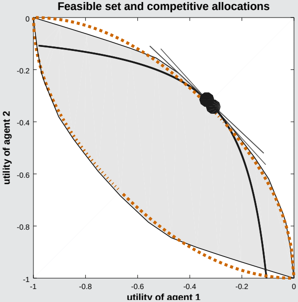
Multiplicity becomes degenerate for large problems

- u_{ia} are i.i.d. random variables uniformly distributed on $[-\frac{1}{m}, 0]$.

Proposition

Two agents divide m bads, $m \to \infty$. Fix $\varepsilon > 0$. Utility vectors of all competitive allocations are concentrated in ε-neighbourhood of $\left(-\frac{1}{3}, -\frac{1}{3}\right)$ with probability $p_m \to 1$.



Example with 15 bads.

Algorithmic questions

Theorem (Vazirani (2006))

The outcome of the MaxNashProduct can be computed in $O\left(poly\left(|N| + |M|\right)\right)$.

Question: Is this true for all-bads problem?

New features:

- critical points (local extrema and saddle points) on the boundary instead of global extremum
- multiplicity

Computing all outcomes

Observation: if M and N are both large $\Rightarrow$ no polynomial algorithm, since the number of outcomes can be exponential

The case of $|N| = 2$

Pareto frontier has simple structure $\Rightarrow$ simple polynomial algorithm.

- Rearrange bads in such a way that $\frac{u_{1a}}{u_{2a}}$ is increasing
- Then any Pareto allocation z has the form

$$z = \begin{pmatrix} 1 & 1 & \dots & 1 & x & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1-x & 1 & 1 & \dots & 1 \end{pmatrix}$$

- For any allocation of this form we can check FOC of criticality

Corollary: there are most $2|M| - 1$ outcomes

Conjecture

The same idea works for arbitrary fixed N: compute Pareto frontier and check every face using FOC.

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Conjecture

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Computing at least one outcome

Open question: When N and M are both large, can a particular outcome of the Competitive Rule be computed in polynomial time (i.e., a *selection*, e.g. MaxMinNashProduct)?

Indivisibilities

For indivisible items the notion of envy-freeness should be relaxed to guarantee existence.

Envy-Free-1 allocations for goods (Budish 2011)

Allocation z of indivisible items is Envy-Free-1 iff

$$\forall i, j \in N \quad \exists a \in z_j : \quad u_i(z_i) \geq u_i(z_j \setminus \{a\}).$$

Theorem (Caragiannis et al. (2016))

For goods, maximization of Nash Social Welfare over indivisible allocations leads to Efficient Envy-Free-1 allocation.

Open question: Do Efficient Envy-Free-1 allocations exist for bads?

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Conclusions

Concluding remarks:

- First results on mixed problem (goods + bads)
- All-bads problem differs from all-goods
- The MaxNashProduct rule can be extended to mixed problems; it is still appealing but becomes multivalued for all-bads case
- Computing the outcome of MaxNashProduct for bads is no longer a convex optimization problem

Future research:

- Algorithms
- Selectors
- Indivisibilities

Thank you!

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$(\text{Thank you!})^2$