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Israel Institute
of Technology

HIGHER SCHOOL OF ECONOMICS
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FAIR DIVISION WITH MINIMAL SHARING

PROBLEM OF FAIR DIVISION

- **n agents** with **different preferences** over **m goods**
- **The goal:** find **«Fair**» & **Pareto Optimal (PO) allocation, no money transfers**
  - **Applications:** dissolving partnership (e.g., divorce), tasks or offices space to workers, seats at over-demanded courses, public housing, charity

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CHALLENGE FOR THEORY: INDIVISIBILITY

- **Microeconomic theory works with divisible resources**
  - Mathematical easiness
  - Good approximation for supply-demand framework
  - Fair & PO allocations exist (under some assumptions)

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- **Practice: how to divide 3 apartments and 1 car between 2 people?**
    - **Bad news: for indivisible items a fair allocation may fail to exist**
        - **Example:** 1 apartment, 2 agents

HOW TO DEAL WITH INDIVISIBILITIES? EXISTING APPROACHES

- **Microeconomics: let's make them divisible**
  - **Randomization:** 0.5 of a bicycle = getting the whole bicycle with probability 1/2
  - **Time-sharing / co-ownership:** 0.5 of a bicycle = using the bicycle 1/2 of a time

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  - **Envy-freeness up to one good*** / **MaxMinShare**:** Fair & PO allocations exist

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Question: Will you be satisfied by an allocation that is **envy-free up to one apartment**? Gives you an **apartment with probability 1/2**?

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- The **problem is** computationally **hard** (NP-hard) **for 2 agents with identical additive utilities.**
- However, **for any** fixed number of agents **n,** a **random problem is simple with probability 1**.

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- n agents with and m **divisible** goods

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$v =$

			
Alice	5	10	3
Bob	3	8	1

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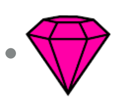
$V_{\text{Alice}}(0\cdot$  $+\frac{1}{2}\cdot$  $+\frac{1}{3}\cdot$  $) = 0\cdot 5+\frac{1}{2}\cdot 10+\frac{1}{3}\cdot 3 = 6$

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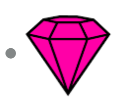
LIKE ON spliddit

$V_{\text{Alice}}(0\cdot$  $+\frac{1}{2}\cdot$  $+\frac{1}{3}\cdot$  $) = 0\cdot 5+\frac{1}{2}\cdot 10+\frac{1}{3}\cdot 3 = 6$

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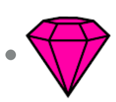
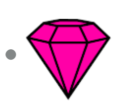
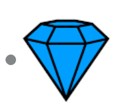
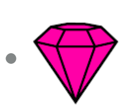
$V_{\text{Alice}}(0 \cdot$  $+ \frac{1}{2} \cdot$  $+ \frac{1}{3} \cdot$  $) = 0 \cdot 5 + \frac{1}{2} \cdot 10 + \frac{1}{3} \cdot 3 = 6$

- allocation $z =$ collection of bundles: all goods are distributed

$z_{\text{Alice}} = 0 \cdot$  $+ \frac{1}{2} \cdot$  $+ \frac{1}{3} \cdot$ 

$z_{\text{Bob}} = 1 \cdot$  $+ \frac{1}{2} \cdot$  $+ \frac{2}{3} \cdot$ 

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Fairness

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$$V_{\text{Alice}}(z_{\text{Alice}}) \geq V_{\text{Alice}}(z_{\text{Bob}})$$

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Equal-Split Lower Bound (ELB) aka Fair Share

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with **minimal number of shared goods**

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Dickerson, Goldman, Karp, Procaccia, Sandholm (2014) *The computational rise and fall of fairness.* AAAI'14

~~DE~~MOTIVATING EXAMPLE

Two agents, Alice and Bob with **identical preferences**

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Alice	w_1	w_2	……	w_{m-1}	w_m
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Pessimistic conclusion*

Checking existence of Fair & PO allocations with no sharing is **hard even for 2 agents with identical preferences.** ***de Keijzer, Bouveret, Klos, Zhang** (2009) *On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences*

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Our simple observation: the problem is hard **only for degenerate preferences** (when the set of PO allocations is exponential)

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Optimistic conclusion

For almost all V with 2 agents, Fair & PO allocation with no sharing can be found (if exists) using $O(m\log(m))$ operations.

MAIN RESULT: n AGENTS

Degree of degeneracy:

$$D(v) = \max_{\text{agents } i \neq j} \max_{r>0} \#\left\{ \text{goods } g \; : \; \frac{v_{i,g}}{v_{j,g}} = r \right\} - 1$$

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- If $D(v) = O(\log(m))$, then finding Fair & PO allocation with minimal sharing **takes polynomial time in m**
- If $D(v) \geq C \cdot m^{\alpha}$ for some $C, \alpha > 0$, then checking existence of Fair & PO allocation with no sharing **is NP-hard**

SKETCH OF THE PROOF FOR $D(v) = 0$

Consumption graph G of an allocation z : bipartite graph on (agents–goods), where i and g are connected if $z_{i,g} > 0$

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Takes constant time since n is fixed!

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Remains to check: all PO consumption graphs can be enumerated in polynomial time for fixed n.

COMPUTING A SET OF GRAPHS CONTAINING G OF ALL PO ALLOCATIONS*

*** Branzei, Sandomirskiy** (2019)
Algorithms for competitive division of chores

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- **n=2: we already know the answer**

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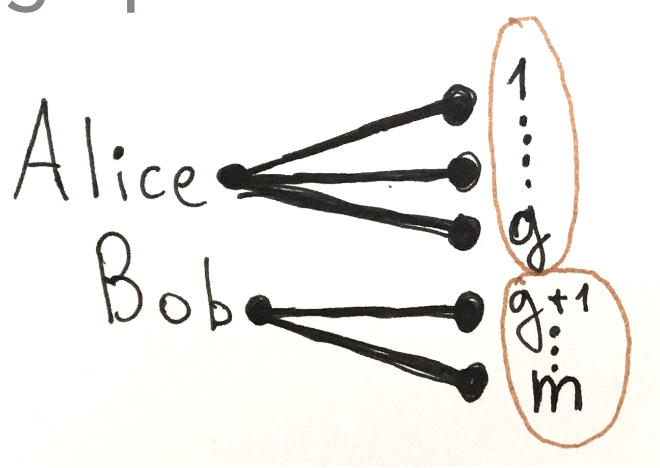
$m+1$ graph with 0 shared goods



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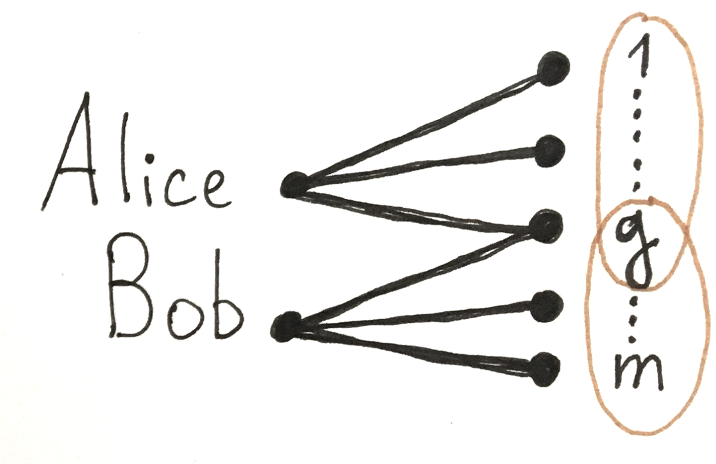

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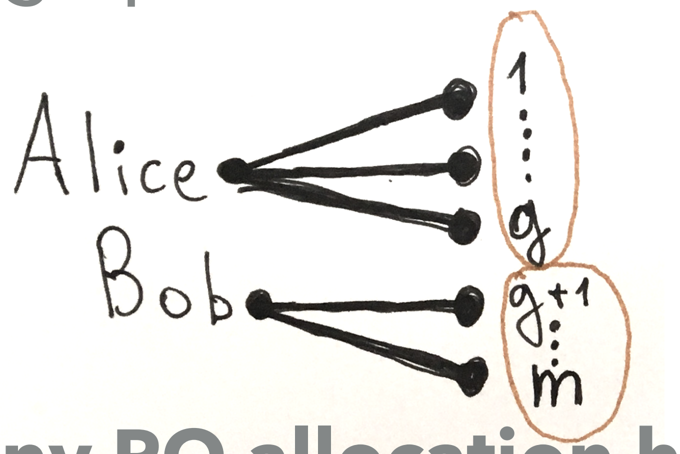

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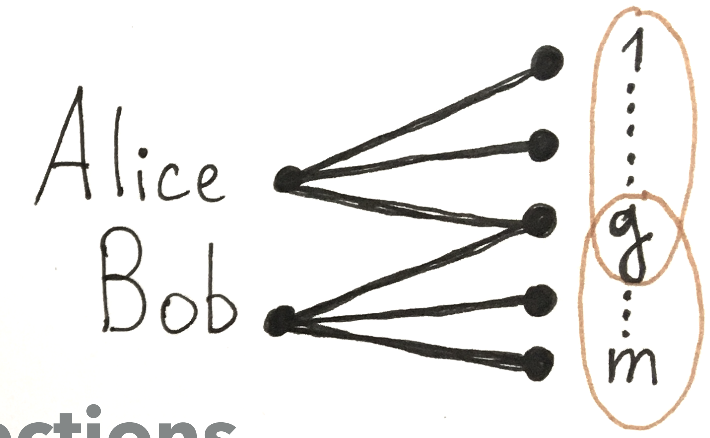

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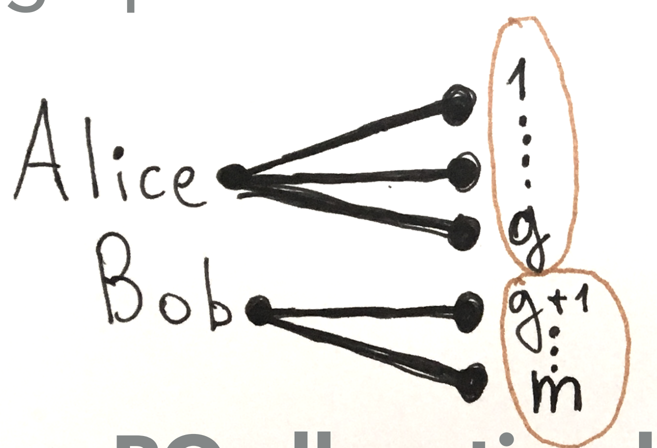

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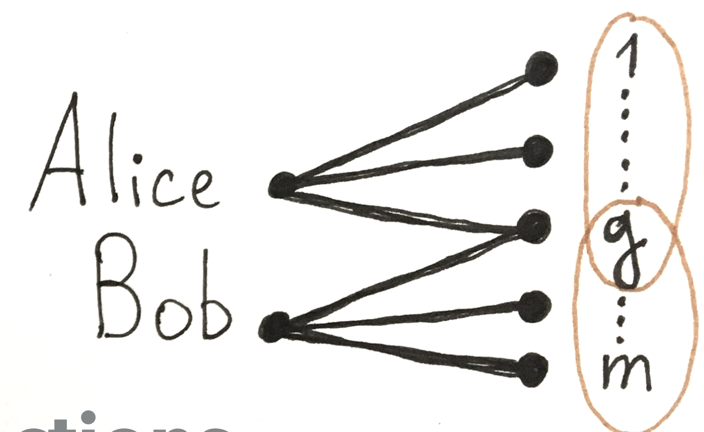

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Corollary: G of any PO allocation can be obtained by

- Picking a PO graph for each pair of agents
  - $(2m+1)^{\frac{n(n-1)}{2}}$ possibilities (polynomial number)
- Tracing an edge between an agent i and a good g if this edge is traced in all 2-agent graphs with i.

CHECKING PARETO OPTIMALITY

The computed set of graphs may contain some **non-PO graphs**. How to detect them?

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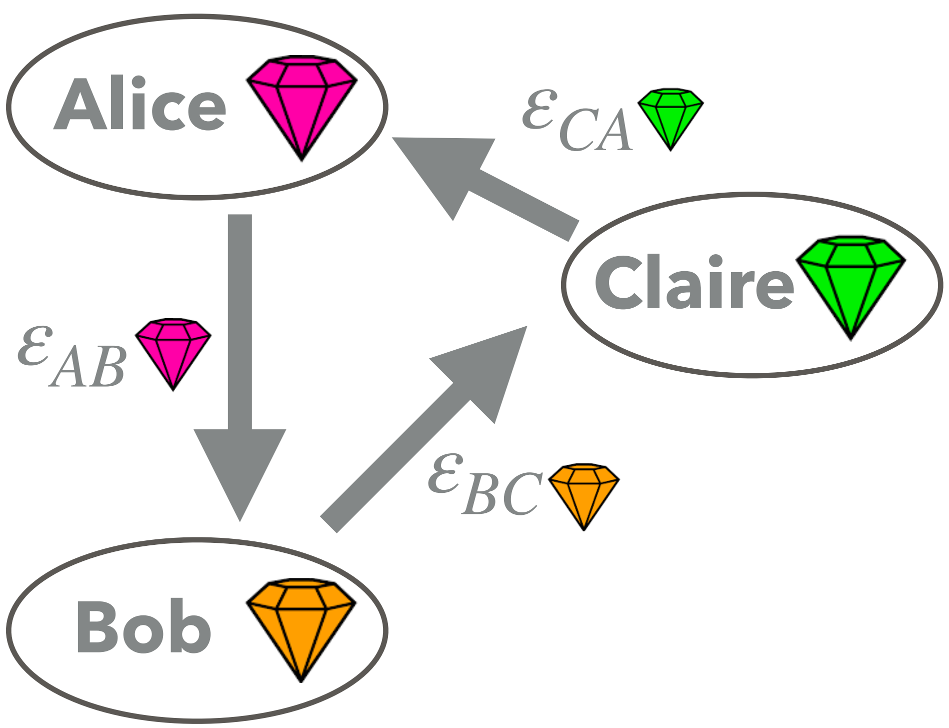

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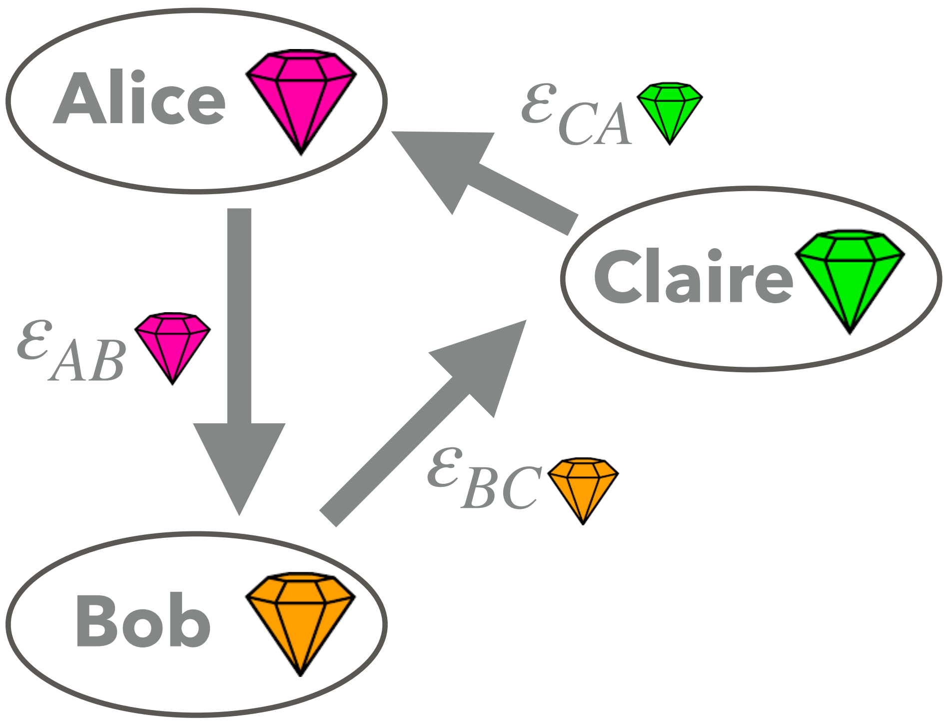

Directed weighted consumption graph $\overrightarrow{G}$:

- edge $(i \rightarrow g)$ is traced if $z_{i,g} > 0$, weight $\log(v_{i,g})$
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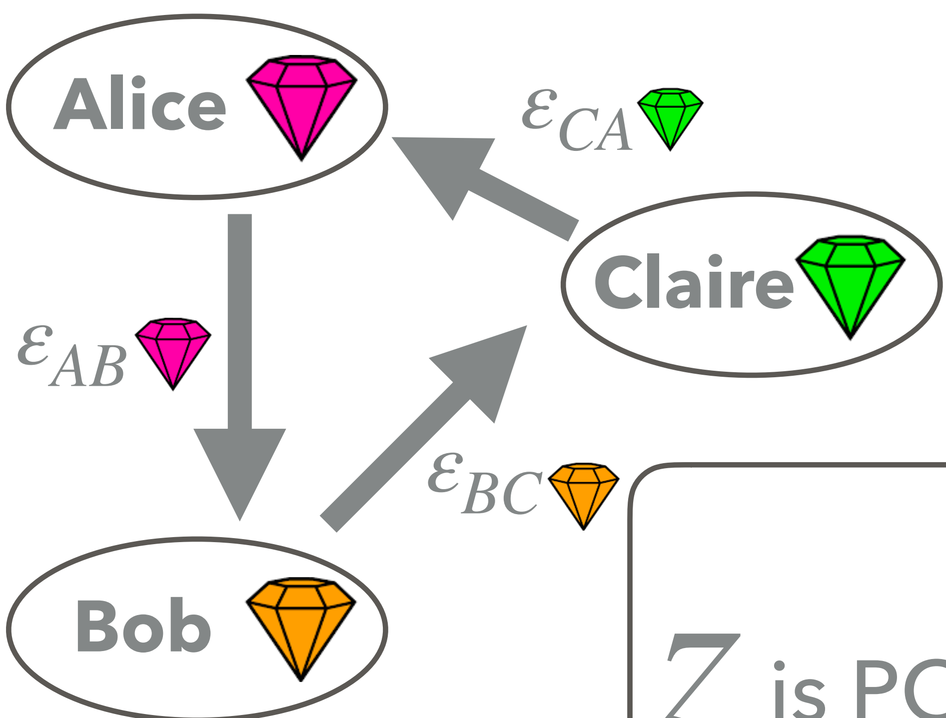
CHECKING PARETO OPTIMALITY

The computed set of graphs may contain some **non-PO graphs**. How to detect them?

Origin of non-optimality = profitable cyclical exchanges:



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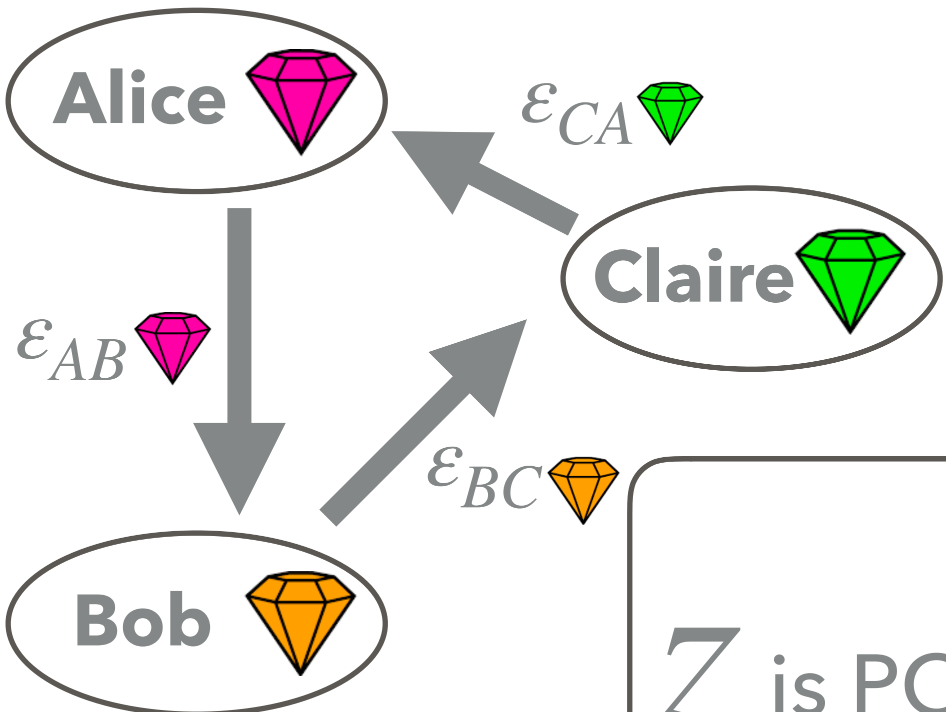

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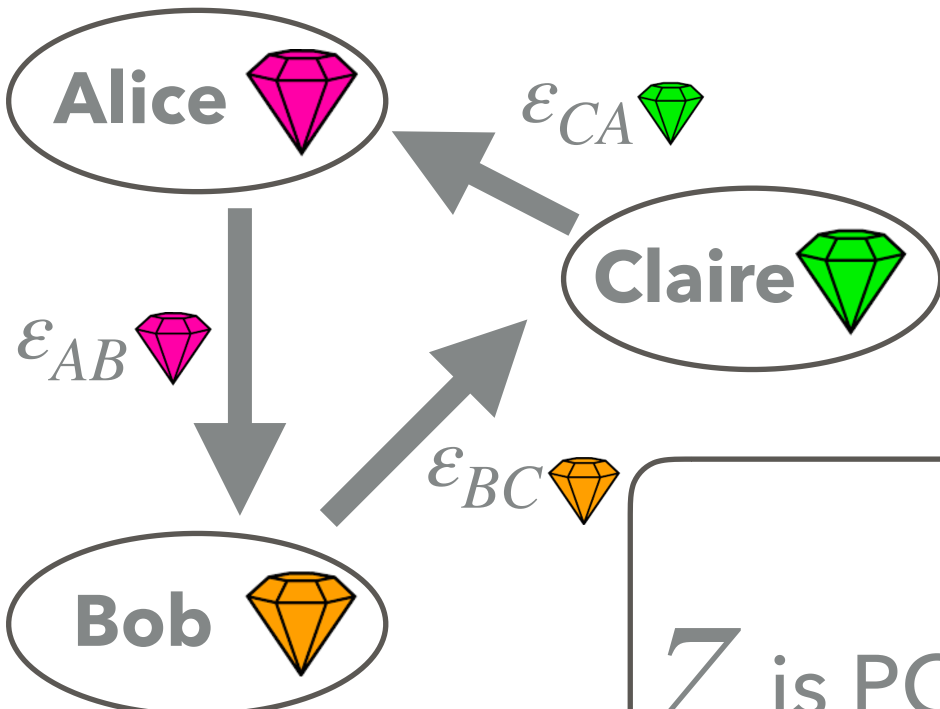

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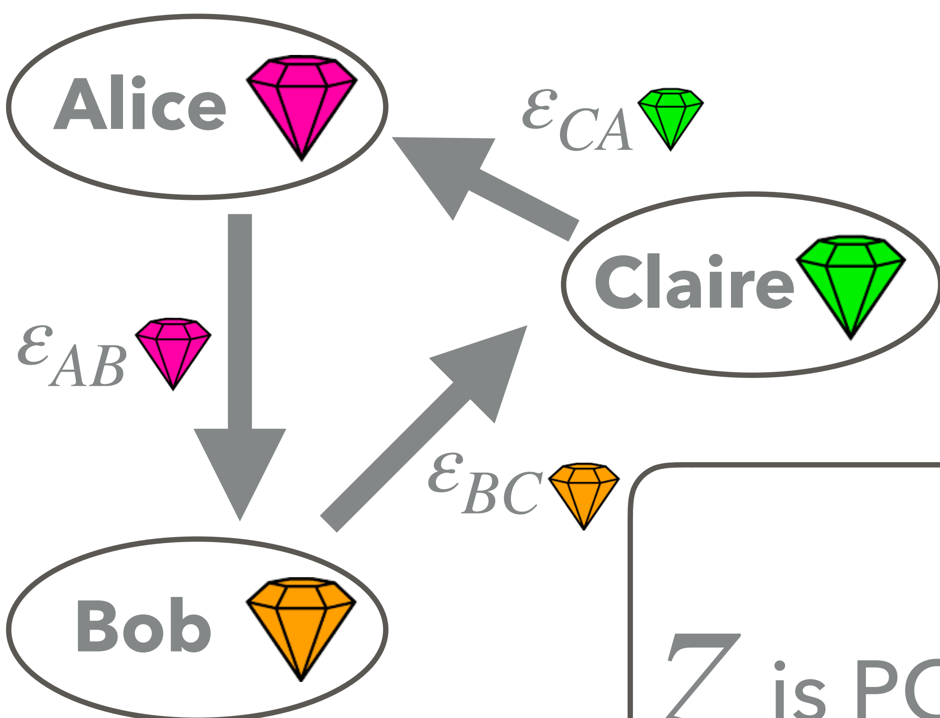

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Usual (fractional) **PO is a better notion****

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**__Barman, Krishnamurthy, Vaish__ (2018). *Finding fair and efficient allocations.*

CONCLUSIONS:

Conceptual

- **Failure of randomization** and **approximate fairness** for **valuable goods**
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- **Typical** problems **are better**-behaved **than worst-cases**
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Thank you!