On social networks that support learning

arXiv:2011.05255

Itai Arieli, **Fedor Sandomirskiy***, Rann Smorodinsky

*Technion, Haifa & Higher School of Economics, St.Petersburg → Caltech
e-mail: fedor.sandomirskiy@gmail.com
homepage: https://www.fedors.info/

Social learning

- each agent is going to make a single decision
  - Android/iPhone, Private/Public kindergartens, restaurant A/B
- gets individual noisy signal about the best alternative & observes choices made by predecessors

Social learning

- each agent is going to make a single decision
  - Android/iPhone, Private/Public kindergartens, restaurant A/B
- gets individual noisy signal about the best alternative & observes choices made by predecessors

- **usually: failure of information aggregation (herding)**
  - first agents take the wrong action $\Rightarrow$ others repeat it & ignore their private signals $\Rightarrow$ information cascade (Banerjee [1992], Bikhchandani et al. [1992])

Social learning

- each agent is going to make a single decision
  - Android/iPhone, Private/Public kindergartens, restaurant A/B
- gets individual noisy signal about the best alternative & observes choices made by predecessors

- **usually: failure of information aggregation (herding)**
  - first agents take the wrong action ⇒ others repeat it & ignore their private signals ⇒ information cascade (Banerjee [1992], Bikhchandani et al. [1992])
- **mitigation measures**
  - signals of unbounded quality (Smith and Sorensen [2000])
  - restricted observation: actions of friends on a social network (Smith [1991], Sgroi [2002], Acemoglu et al. [2010])

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
    - cannot stop the information cascade

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
    - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
  - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?**

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
  - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?** NO and YES
  - NO
    - topological conditions for a given ordering of agents Smith [1991], Sgroi [2002], Acemoglu et al. [2010]
    - the timing of decisions determines social connections
    - reasonable for life-long decisions (doctor/teacher) but not for (Android/iPhone)

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
  - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?** NO and YES
  - NO
    - topological conditions for a given ordering of agents Smith [1991], Sgroi [2002], Acemoglu et al. [2010]
    - the timing of decisions determines social connections
    - reasonable for life-long decisions (doctor/teacher) but not for (Android/iPhone)
  - YES if the social structure and the timing of decisions are unrelated

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
    - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?** NO and YES
    - NO
        - topological conditions for a given ordering of agents Smith [1991], Sgroi [2002], Acemoglu et al. [2010]
        - the timing of decisions determines social connections
        - reasonable for life-long decisions (doctor/teacher) but not for (Android/iPhone)
    - YES if the social structure and the timing of decisions are unrelated
- **Our model: the network is given and the order is random**

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
    - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?** NO and YES
    - NO
        - topological conditions for a given ordering of agents Smith [1991], Sgroi [2002], Acemoglu et al. [2010]
        - the timing of decisions determines social connections
        - reasonable for life-long decisions (doctor/teacher) but not for (Android/iPhone)
    - YES if the social structure and the timing of decisions are unrelated
- **Our model: the network is given and the order is random**
    - the network must aggregate information for most orders (very demanding!)

Our question

- Agents are Bayesian-rational, sit on a network, act only once
- Signals have bounded quality
    - cannot stop the information cascade

The big puzzle

Which properties of the network are responsible for information aggregation?

- **Are we the first to study this question?** NO and YES
    - NO
        - topological conditions for a given ordering of agents Smith [1991], Sgroi [2002], Acemoglu et al. [2010]
        - the timing of decisions determines social connections
        - reasonable for life-long decisions (doctor/teacher) but not for (Android/iPhone)
    - YES if the social structure and the timing of decisions are unrelated
- **Our model: the network is given and the order is random**
    - the network must aggregate information for most orders (very demanding!)
    - an example of such a network (Bahar et al. [2020])

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away
  - no global information cascades
  - quality of agent's decision is determined by his small neighborhood

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away ⇒
    - no global information cascades
    - quality of agent's decision is determined by his small neighborhood

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away $\Rightarrow$
  - no global information cascades
  - quality of agent's decision is determined by his small neighborhood

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away $\Rightarrow$
  - no global information cascades
  - quality of agent's decision is determined by his small neighborhood
- **Local learning requirement:** the condition on agent's neighborhood for high-quality decision
  - Want well-informed decisions? Make sure to be a part of mutually exclusive social circles!

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away $\Rightarrow$
  - no global information cascades
  - quality of agent's decision is determined by his small neighborhood
- **Local learning requirement:** the condition on agent's neighborhood for high-quality decision
  - Want well-informed decisions? Make sure to be a part of mutually exclusive social circles!

What will we see?

- **Localization phenomenon:** agent's decision is almost independent from those who are far away $\Rightarrow$
  - no global information cascades
  - quality of agent's decision is determined by his small neighborhood
- **Local learning requirement:** the condition on agent's neighborhood for high-quality decision
  - Want well-informed decisions? Make sure to be a part of mutually exclusive social circles!
- **Applications:** constructing networks where learning is robust to disruptions

The model and examples

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
  - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
  - the set of friends who arrived earlier
  - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
  - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
  - the set of friends who arrived earlier
  - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
    - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
    - the set of friends who arrived earlier
    - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
    - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
    - the set of friends who arrived earlier
    - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices $=$ agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
  - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
  - the set of friends who arrived earlier
  - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
    - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
    - the set of friends who arrived earlier
    - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
    - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
    - the set of friends who arrived earlier
    - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

- A finite Bayesian game $\Rightarrow$ an equilibrium exists

The model

The model

- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
  - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
  - the set of friends who arrived earlier
  - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

- A finite Bayesian game $\Rightarrow$ an equilibrium exists
- **Learning qualities of an agent / of the network:**

$$\mathbb{P}(a_v = \theta) \quad \Big/ \quad L(G) = \frac{1}{|V|} \sum_{v \in V} \mathbb{P}(a_v = \theta)$$

The model

The model

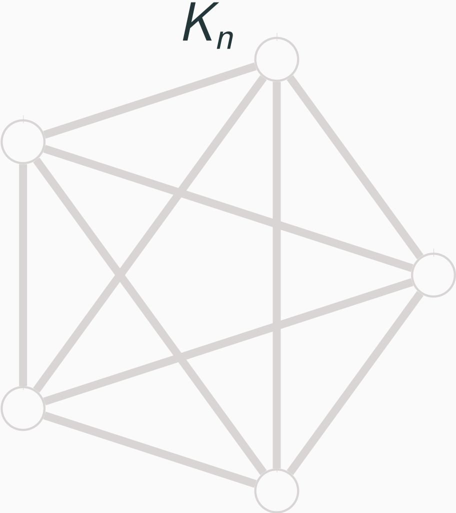
- undirected finite network $G = (V, E)$, vertices = agents
- unobservable state $\theta \in \{\text{blue}, \text{red}\}$ equally likely
- agent $v \in V$ arrives at t_v, i.i.d. uniform on $[0, 1]$
- v takes an action $a_v \in \{\text{blue}, \text{red}\}$ depending on his information
    - a binary signal that matches θ w.p. $p > \frac{1}{2}$ (i.i.d. conditional on θ)
    - the set of friends who arrived earlier
    - their actions
- the utility is 1 if $a_v = \theta$ and 0, otherwise.

- A finite Bayesian game $\Rightarrow$ an equilibrium exists
- **Learning qualities of an agent / of the network:**

$$\mathbb{P}(a_v = \theta) \quad / \quad L(G) = \frac{1}{|V|} \sum_{v \in V} \mathbb{P}(a_v = \theta)$$

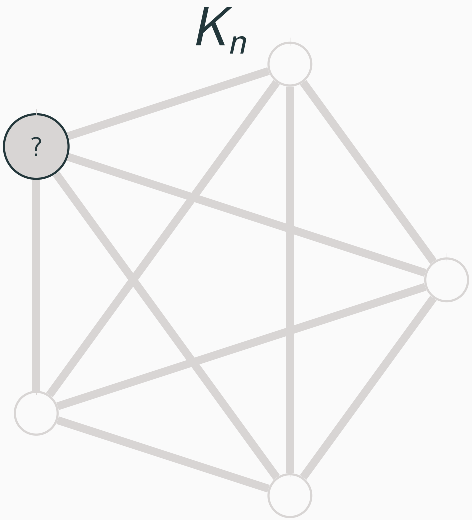
- the network supports learning if $L(G) \approx 1$

Example: information cascade on a clique



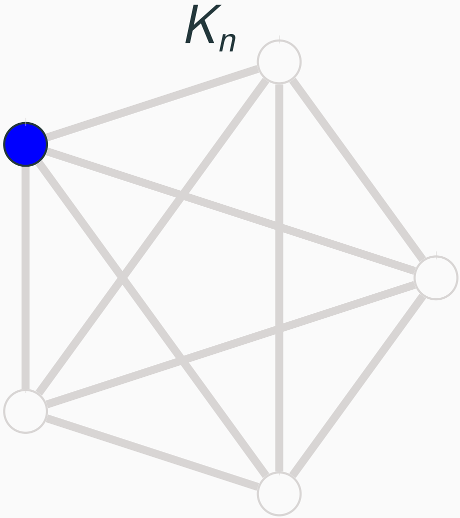
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



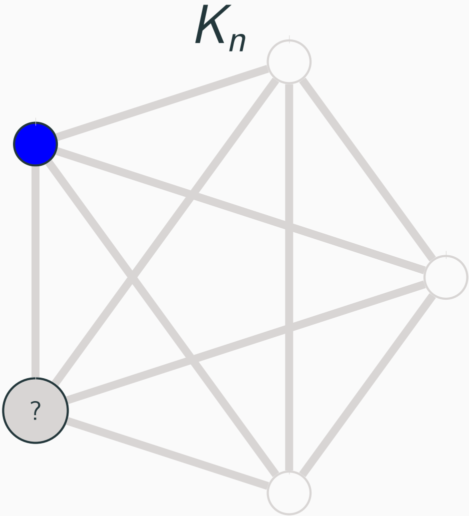
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



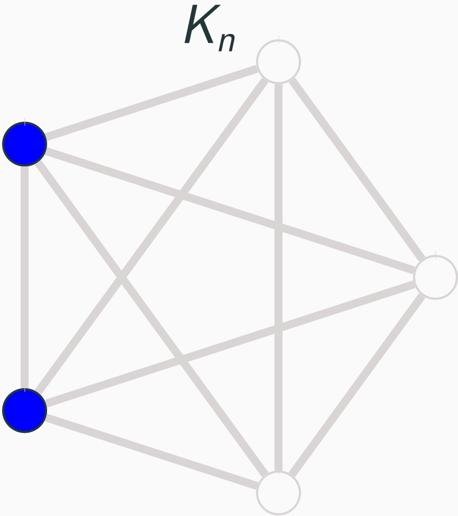
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



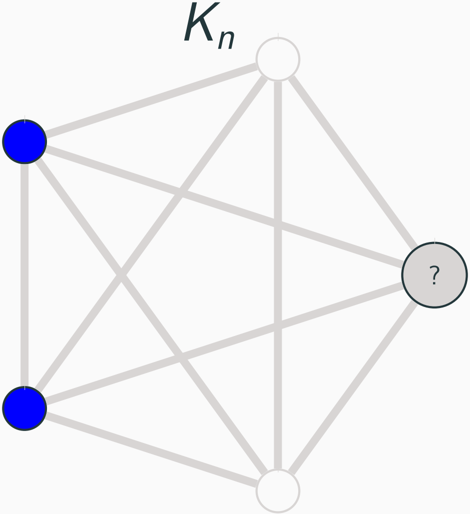
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



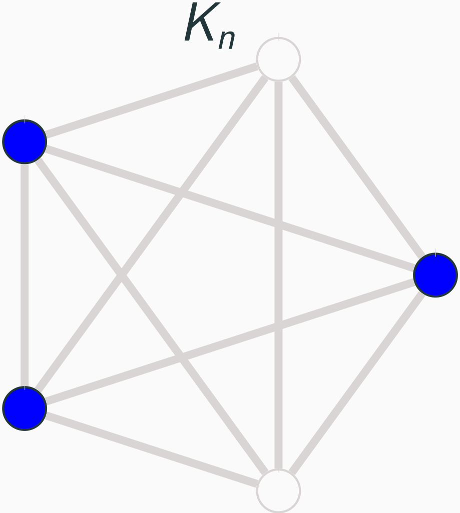
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



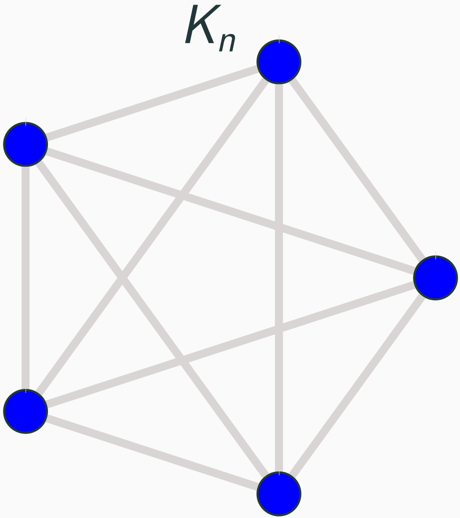
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



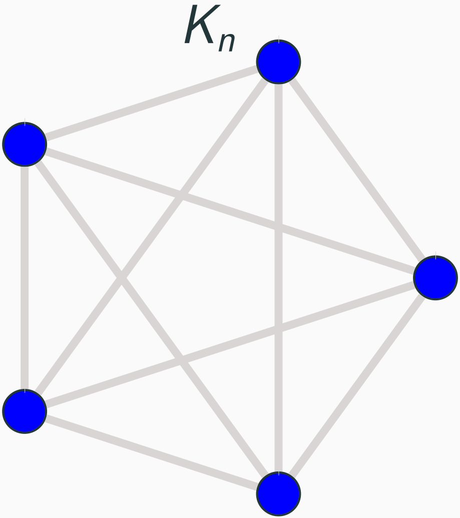
- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

Example: information cascade on a clique



- 1st and 2nd agents get the same blue signals
- 3rd agent repeats their action and ignores his signal
- and so on

The 1st two agents got wrong signals w.p. $(1-p)^2 \Rightarrow$

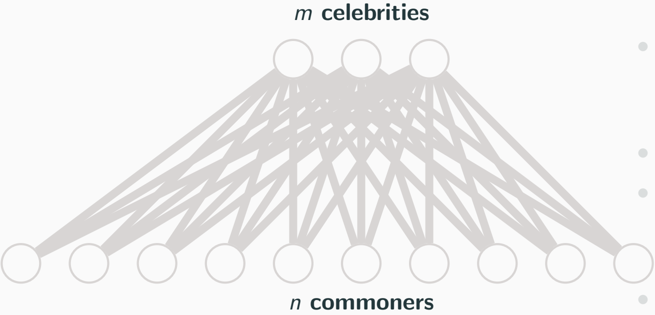
$$L(K_n) \leq 1-(1-p)^2$$

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



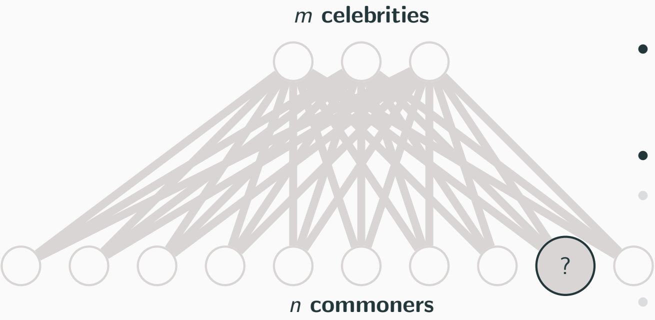
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



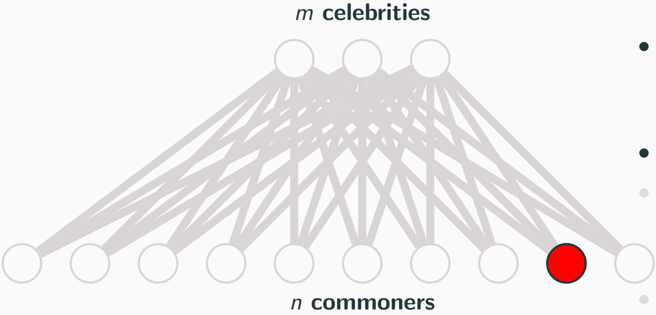
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



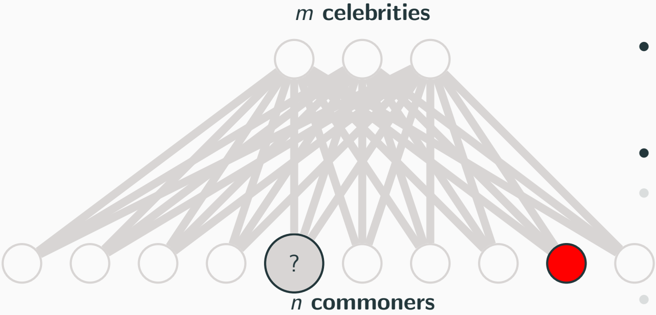
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



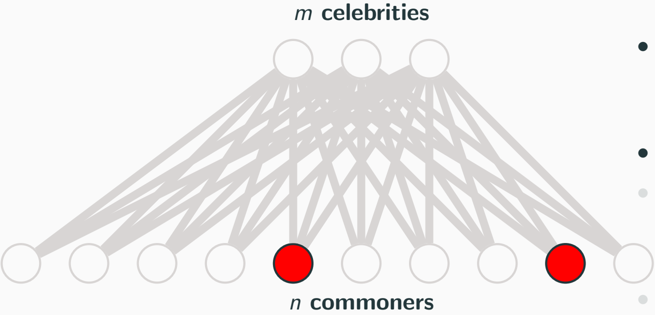
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



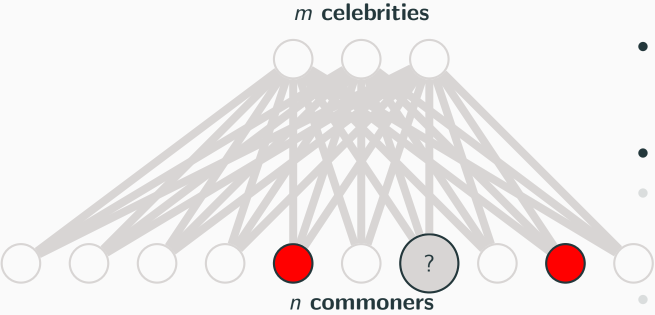
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



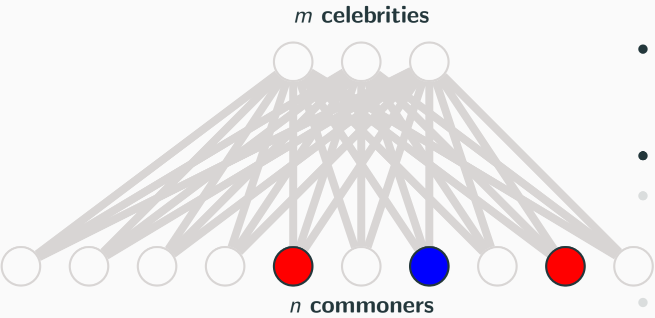
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



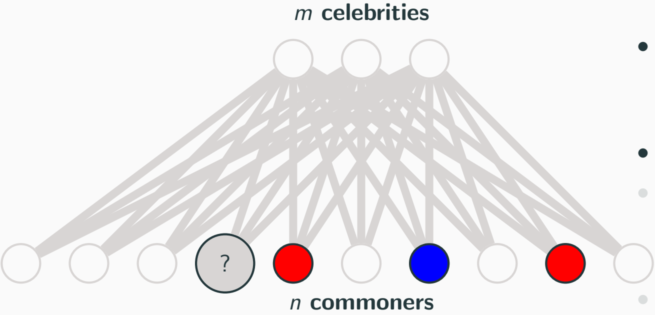
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



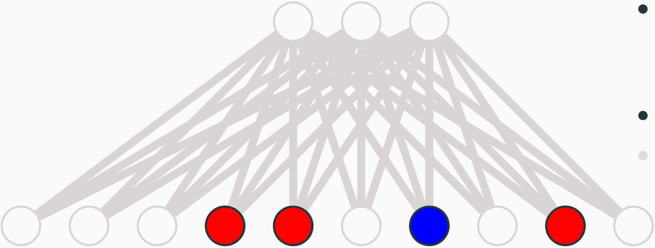
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



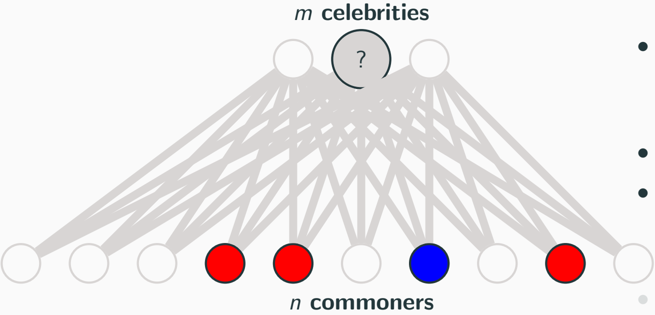
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



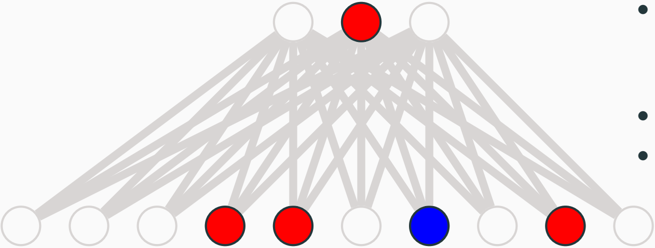
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



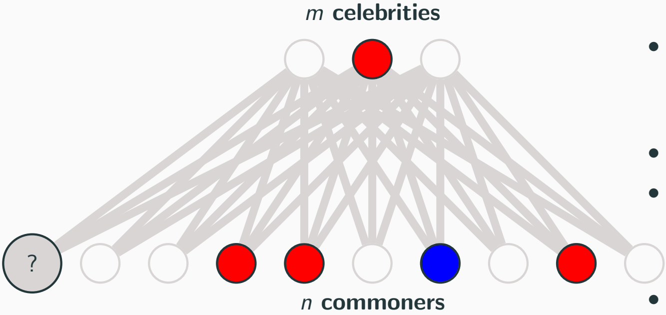
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



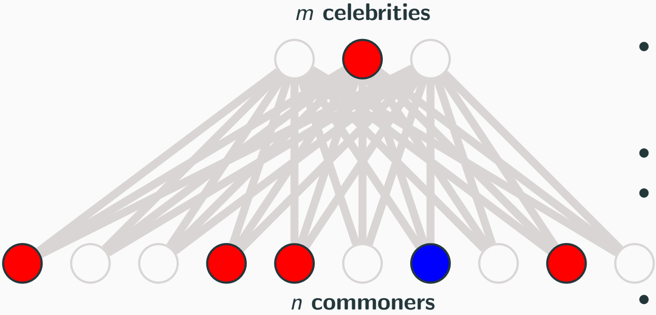
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



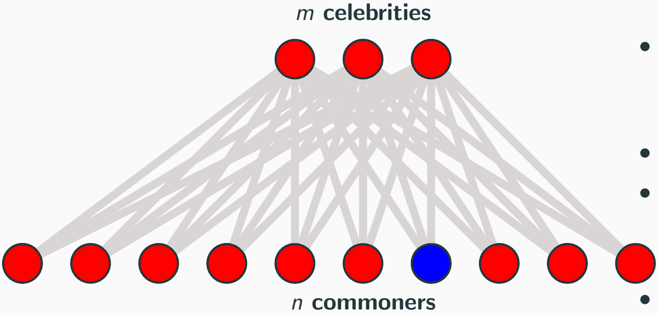
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



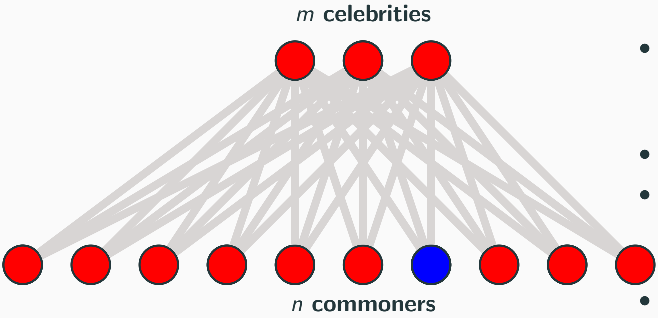
- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Example: celebrity graphs (Bahar et al. [2020])

n commoners and m celebrities observing each other: $n \gg m \gg 1$.

Theorem (Bahar et al. [2020])

$\forall \delta > 0$ there is a celebrity graph with $L \geq 1 - \delta$.



- $\simeq \frac{n}{m} \gg 1$ commoners arrive before the 1st celebrity
- follow their signals
- 1st celebrity aggregates these i.i.d. inputs
- learning propagates

Remarks:

- the only known family of graphs with L close to 1
- non-robustness: minority of celebrities is critical for learning

Our results

Localization phenomenon

Question: When can the action of u affect the action of v?

Localization phenomenon

Question: When can the action of u affect the action of v?

- v observes u, i.e., $vu \in E$ and $t_v > t_u$

Localization phenomenon

Question: When can the action of u affect the action of v?

- v observes u, i.e., $vu \in E$ and $t_v > t_u$
- v observes v_1 who observes u

Localization phenomenon

Question: When can the action of u affect the action of v?

- v observes u, i.e., $vu \in E$ and $t_v > t_u$
- v observes v_1 who observes u
- v observes v_1 who observes v_2 who observes u
- ...

Localization phenomenon

Question: When can the action of u affect the action of v?

- v observes u, i.e., $vu \in E$ and $t_v > t_u$
- v observes v_1 who observes u
- v observes v_1 who observes v_2 who observes u
- $\ldots$
- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \;:\; \exists \text{ a realized path } (v \to u)\}$$

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \dots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \;:\; \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \dots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \;:\; \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Proof: Show that no path $\big(v \to \text{boundary of } r\text{-neighborhood}\big)$ is realized

- a path of length r is realized with probability $1/(r+1)!$
- $\#\{\text{paths of length } r\} \leq D^r$
- the union bound

□

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \; : \; \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Proof: Show that no path $\big(v \to \text{boundary of } r\text{-neighborhood}\big)$ is realized

- a path of length r is realized with probability $1/(r+1)!$
- $\#\{\text{paths of length } r\} \leq D^r$
- the union bound

□

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \;:\; \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Proof: Show that no path $\big(v \to \text{boundary of } r\text{-neighborhood}\big)$ is realized

- a path of length r is realized with probability $1/(r+1)!$
- $\#\{\text{paths of length } r\} \leq D^r$
- the union bound

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \ : \ \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Proof: Show that no path $\big(v \to \text{boundary of } r\text{-neighborhood}\big)$ is realized

- a path of length r is realized with probability $1/(r+1)!$
- $\#\{\text{paths of length } r\} \leq D^r$
- the union bound $\square$

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \ : \ \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Implications:

- $d(v,u) \gg e \cdot D \Longrightarrow \mathbb{P}(a_v \text{ and } a_u \text{ are dependent})$ is exp. small
- impossibility of global information cascades
- the quality of v's decision is determined by the local structure of the network around v

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\text{real}}(v) = \{u \;:\; \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\text{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Implications:

- $d(v, u) \gg e \cdot D \Longrightarrow \mathbb{P}(a_v \text{ and } a_u \text{ are dependent})$ is exp. small
- impossibility of global information cascades
- the quality of v's decision is determined by the local structure of the network around v

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \ : \ \exists \text{ a realized path } (v \to u)\}$$

Proposition

$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Implications:

- $d(v,u) \gg e \cdot D \Longrightarrow \mathbb{P}(a_v \text{ and } a_u \text{ are dependent})$ is exp. small
- impossibility of global information cascades
- the quality of v's decision is determined by the local structure of the network around v

Localization phenomenon

Question: When can the action of u affect the action of v?

- $\exists$ a path $(v = v_0, v_1, \ldots, v_{n-1}, v_n = u)$ such that $t_{v_i} > t_{v_{i+1}} \quad \forall i$
- call such a path **realized**

Definition: Realized component

$$N^{\mathrm{real}}(v) = \{u \ : \ \exists \text{ a realized path } (v \to u)\}$$

Proposition

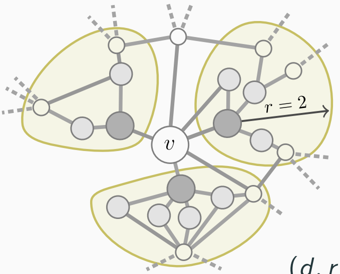
$$\mathbb{P}\Big(N^{\mathrm{real}}(v) \subset r\text{-neighborhood of } v\Big) \geq 1 - 2\left(\frac{e \cdot D}{r}\right)^r,$$

where D is the maximal degree.

Implications:

- $d(v,u) \gg e \cdot D \Longrightarrow \mathbb{P}(a_v \text{ and } a_u \text{ are dependent})$ is exp. small
- impossibility of global information cascades
- the quality of v's decision is determined by the local structure of the network around v

Local Learning Requirement

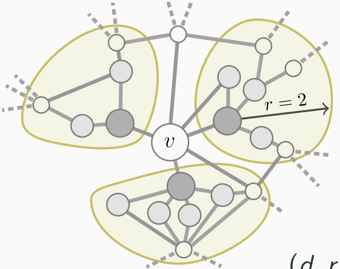


LLR with parameters (d, r, D):

- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

Local Learning Requirement



LLR with parameters (d, r, D):

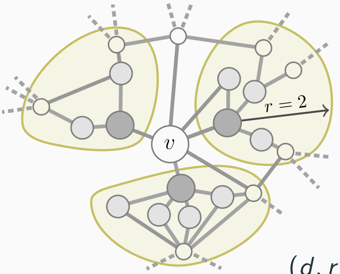
- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Local Learning Requirement



$(d, r, D) = (3, 2, 7)$

LLR with parameters (d, r, D):

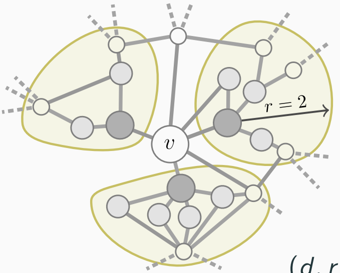
- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Proof:

Local Learning Requirement



LLR with parameters (d, r, D):

- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

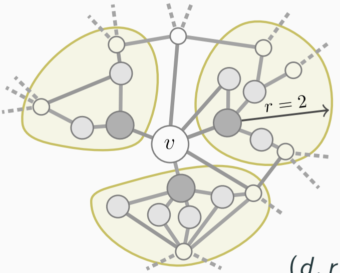
Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Proof:

- **localization** $\Rightarrow$ realized components of v's friends are disjoint with probability $\geq 1 - \psi \Rightarrow$ **independence**

Local Learning Requirement



LLR with parameters (d, r, D):

- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

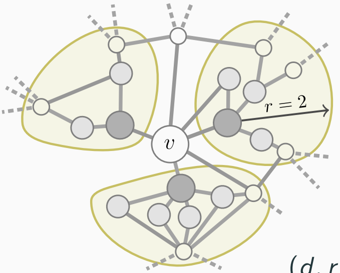
Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Proof:

- **localization** $\Rightarrow$ realized components of v's friends are disjoint with probability $\geq 1 - \psi \Rightarrow$ **independence**
- each friend takes correct action with prob. $\geq p \Rightarrow$ **informativeness**

Local Learning Requirement



LLR with parameters (d, r, D):

- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

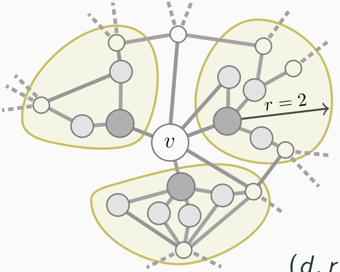
Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Proof:

- **localization** $\Rightarrow$ realized components of v's friends are disjoint with probability $\geq 1 - \psi \Rightarrow$ **independence**
- each friend takes correct action with prob. $\geq p \Rightarrow$ **informativeness**
- v observes $O(d)$ independent sources $\Rightarrow$ use **Chernoff's bound**. $\square$

Local Learning Requirement



LLR with parameters (d, r, D):

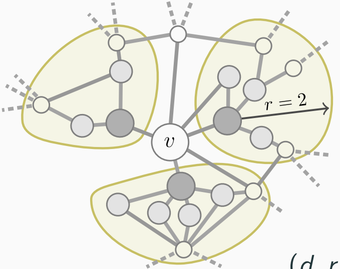
- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Local Learning Requirement



LLR with parameters (d, r, D):

- v has a subset of $\geq d$ friends s.t.
  - each of them has degree $\geq d$
  - their r-neighborhoods in $G \setminus v$ are disjoint
  - the max degree in these neighborhoods $\leq D$

$(d, r, D) = (3, 2, 7)$

Theorem

$$\mathbb{P}(a_v = \theta) \geq 1 - \left(\psi + \frac{18}{\sqrt{d-1}\,(2p-1-\psi)}\right), \quad \text{where} \quad \psi = r \cdot \left(\frac{e \cdot D}{r}\right)^r$$

Global implications of LLR: apply to each agent in the network

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\star)$$

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \quad (\bigstar)$$

Why surprising?

- theory of the two-step information flow (Katz and Lazarsfeld [1955]): $\exists$ a minority critical for information-aggregation and predetermined by the network structure (opinion leaders)
  - e.g., celebrities in Bahar et al. [2020]: if eliminated $\Rightarrow$ no aggregation
- Bayesian social learning is fragile Frick et al. [2020], Mueller-Frank [2018], Bohren [2016]

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Why surprising?

- theory of the two-step information flow (Katz and Lazarsfeld [1955]): $\exists$ a minority critical for information-aggregation and predetermined by the network structure (opinion leaders)
  - e.g., celebrities in Bahar et al. [2020]: if eliminated $\Rightarrow$ no aggregation
- Bayesian social learning is fragile Frick et al. [2020], Mueller-Frank [2018], Bohren [2016]

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Why surprising?

- theory of the two-step information flow (Katz and Lazarsfeld [1955]): $\exists$ a minority critical for information-aggregation and predetermined by the network structure (opinion leaders)
  - e.g., celebrities in Bahar et al. [2020]: if eliminated $\Rightarrow$ no aggregation
- Bayesian social learning is fragile Frick et al. [2020], Mueller-Frank [2018], Bohren [2016]

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Why surprising?

- theory of the two-step information flow (Katz and Lazarsfeld [1955]): $\exists$ a minority critical for information-aggregation and predetermined by the network structure (opinion leaders)
  - e.g., celebrities in Bahar et al. [2020]: if eliminated $\Rightarrow$ no aggregation
- Bayesian social learning is fragile Frick et al. [2020], Mueller-Frank [2018], Bohren [2016]

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L(G|_U) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Why surprising?

- theory of the two-step information flow (Katz and Lazarsfeld [1955]): $\exists$ a minority critical for information-aggregation and predetermined by the network structure (opinion leaders)
  - e.g., celebrities in Bahar et al. [2020]: if eliminated $\Rightarrow$ no aggregation
- Bayesian social learning is fragile Frick et al. [2020], Mueller-Frank [2018], Bohren [2016]

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Proof:

- Apply LLR to each agent in G
  - $(\bigstar)$: G is symmetric, high degrees, no short cycles
  - $(\bigstar\bigstar)$: additionally, most $u \in U$ have high degrees in $G|_U$ for $U \subset V$
- existence of such $G \Longleftarrow$ theory of expanders more details □

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Proof:

- Apply LLR to each agent in G
  - $(\bigstar)$: G is symmetric, high degrees, no short cycles
  - $(\bigstar\bigstar)$: additionally, most $u \in U$ have high degrees in $G|_U$ for $U \subset V$
- existence of such $G \Longleftarrow$ theory of expanders more details □

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Proof:

- Apply LLR to each agent in G
  - $(\bigstar)$: G is symmetric, high degrees, no short cycles
  - $(\bigstar\bigstar)$: additionally, most $u \in U$ have high degrees in $G|_U$ for $U \subset V$
- existence of such $G \Longleftarrow$ theory of expanders more details □

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\bigstar)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\bigstar\bigstar)$$

Proof:

- Apply LLR to each agent in G
  - $(\bigstar)$: G is symmetric, high degrees, no short cycles
  - $(\bigstar\bigstar)$: additionally, most $u \in U$ have high degrees in $G|_U$ for $U \subset V$
- existence of such $G \Longleftarrow$ theory of expanders more details □

Applications: egalitarian societies and robust learning

Symmetry: $G = (V, E)$ is symmetric if for any $v, v' \in V$, there is an automorphism f such that $f(v) = v'$.

Proposition

For any $\delta > 0$ there exists a symmetric network $G = (V, E)$ with

$$L(G) \geq 1 - \delta. \qquad (\star)$$

Moreover, for any $U \subset V$, the sub-network satisfies

$$L\big(G|_U\big) \geq 1 - \frac{\delta}{\alpha^3}, \qquad \text{where} \quad \alpha = \frac{|U|}{|V|}. \qquad (\star\star)$$

Proof:

- Apply LLR to each agent in G
  - $(\star)$: G is symmetric, high degrees, no short cycles
  - $(\star\star)$: additionally, most $u \in U$ have high degrees in $G|_U$ for $U \subset V$
- existence of such $G \Longleftarrow$ theory of expanders more details □

Summary

- **Decoupling the network and the order of actions**
    - long paths of information transmission & global cascades are unlikely
    - learning quality of an agent is determined by the local structure
    - LLR: a necessary condition for high quality & no local cascades
- **Bayesian models do not have explicit solutions**
    - Our approach is indirect. No insights in how equilibria look like.
- **Future:**
    - How do equilibria look like? a simple open problem
    - Other necessary and sufficient conditions for high learning quality

Summary

- **Decoupling the network and the order of actions**
  - long paths of information transmission & global cascades are unlikely
  - learning quality of an agent is determined by the local structure
  - LLR: a necessary condition for high quality & no local cascades
- **Bayesian models do not have explicit solutions**
  - Our approach is indirect. No insights in how equilibria look like.
- **Future:**
  - How do equilibria look like? a simple open problem
  - Other necessary and sufficient conditions for high learning quality

Thank you!

References

Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar. Bayesian learning in social networks. Review of Economic Studies, 78: 1–34, 2010.

Noga Alon and Fan RK Chung. Explicit construction of linear sized tolerant networks. Discrete Mathematics, 72(1-3):15–19, 1988.

Gal Bahar, Itai Arieli, Rann Smorodinsky, and Moshe Tennenholtz. Multi-issue social learning. Mathematical Social Sciences, 104:29–39, 2020.

Abhijit V Banerjee. A simple model of herd behavior. The quarterly journal of economics, 107(3):797–817, 1992.

S. Bikhchandani, D. Hirshleifer, and I. Welch. A theory of fads, fashion, custom and cultural change as information cascade. The Journal of Political Economy, 100:992–1026, 1992.

J Aislinn Bohren. Informational herding with model misspecification. Journal of Economic Theory, 163:222–247, 2016.

Xavier Dahan. Regular graphs of large girth and arbitrary degree. Combinatorica, 34(4):407–426, 2014.

Mira Frick, Ryota Iijima, and Yuhta Ishii. Misinterpreting others and the fragility of social learning. Econometrica (forthcoming), 2020.

Elihu Katz and Paul F Lazarsfeld. Personal influence: the part played by people in the flow of mass communications. 1955.

Alexander Lubotzky, Ralph Phillips, and Peter Sarnak. Ramanujan graphs. Combinatorica, 8(3):261–277, 1988.

Manuel Mueller-Frank. Manipulating opinions in social networks. Available at SSRN 3080219, 2018.

Daniel Sgroi. Optimizing information in the herd: Guinea pigs, profits, and welfare. Games and Economic Behavior, 39:137–166, 2002.

L. Smith and P. Sorensen. Pathalogical outcomes of observational learning. Econometrica, 68:371–398, 2000.

Lones A Smith. Essays on dynamic models of equilibrium and learning. PhD thesis, University of Chicago, Department of Economics, 1991.

Why expander graphs? back to applications

We need $G = (V, E)$ such that:

- $(\bigstar)$ symmetric & minimal degree is high & no short cycles
- $(\bigstar\bigstar)$ most $u \in U$ have high degrees in $G|_U$, $\forall U \subset V$ big enough

Definition: d-regular graph G is an **expander** if $\lambda_2(G) \ll \lambda_1(G) = d$.

- the simple random walk forgets the origin fast
- $\Rightarrow$ best expanders have no short cycles and are highly connected

Why expander graphs? back to applications

We need $G = (V, E)$ such that:

- ($\bigstar$) symmetric & minimal degree is high & no short cycles
- ($\bigstar\bigstar$) most $u \in U$ have high degrees in $G|_U$, $\forall U \subset V$ big enough

Definition: d-regular graph G is an **expander** if $\lambda_2(G) \ll \lambda_1(G) = d$.

- the simple random walk forgets the origin fast
- $\Rightarrow$ best expanders have no short cycles and are highly connected

Ramanujan expanders (Lubotzky et al. [1988], Dahan [2014])

$\forall d \geq 11$ and $\forall g \geq 0$ $\exists$ symmetric G with cycles $\geq g$ and $\lambda_2 \leq 2\sqrt{d-1}$

Why expander graphs? back to applications

We need $G = (V, E)$ such that:

- ($\bigstar$) symmetric & minimal degree is high & no short cycles
- ($\bigstar\bigstar$) most $u \in U$ have high degrees in $G|_U$, $\forall U \subset V$ big enough

Definition: d-regular graph G is an **expander** if $\lambda_2(G) \ll \lambda_1(G) = d$.

- the simple random walk forgets the origin fast
- $\Rightarrow$ best expanders have no short cycles and are highly connected

Ramanujan expanders (Lubotzky et al. [1988], Dahan [2014])

$\forall d \geq 11$ and $\forall g \geq 0$ $\exists$ symmetric G with cycles $\geq g$ and $\lambda_2 \leq 2\sqrt{d-1}$

For $U, U' \subset V$, denote $E(U, U') = \{e \in E \ : \ e \text{ connects } U \text{ and } U'\}$.

Why expander graphs? back to applications

We need $G = (V, E)$ such that:

- ($\bigstar$) symmetric & minimal degree is high & no short cycles
- ($\bigstar\bigstar$) most $u \in U$ have high degrees in $G|_U$, $\forall U \subset V$ big enough

Definition: d-regular graph G is an **expander** if $\lambda_2(G) \ll \lambda_1(G) = d$.

- the simple random walk forgets the origin fast
- $\Rightarrow$ best expanders have no short cycles and are highly connected

Ramanujan expanders (Lubotzky et al. [1988], Dahan [2014])

$\forall d \geq 11$ and $\forall g \geq 0$ $\exists$ symmetric G with cycles $\geq g$ and $\lambda_2 \leq 2\sqrt{d-1}$

For $U, U' \subset V$, denote $E(U, U') = \{e \in E \,:\, e \text{ connects } U \text{ and } U'\}$.

Mixing lemma (Alon and Chung [1988])

$$\left|E(U, U')\right| = \frac{d}{|V|} \cdot |U| \cdot |U'| + \tau, \quad \text{where} \quad |\tau| \leq \lambda_2 \sqrt{|U||U'|}.$$

Why expander graphs? back to applications

We need $G = (V, E)$ such that:

- ($\bigstar$) symmetric & minimal degree is high & no short cycles
- ($\bigstar\bigstar$) most $u \in U$ have high degrees in $G|_U$, $\forall U \subset V$ big enough

Definition: d-regular graph G is an **expander** if $\lambda_2(G) \ll \lambda_1(G) = d$.

- the simple random walk forgets the origin fast
- $\Rightarrow$ best expanders have no short cycles and are highly connected

Ramanujan expanders (Lubotzky et al. [1988], Dahan [2014])

$\forall d \geq 11$ and $\forall g \geq 0$ $\exists$ symmetric G with cycles $\geq g$ and $\lambda_2 \leq 2\sqrt{d-1}$

For $U, U' \subset V$, denote $E(U, U') = \{e \in E \; : \; e \text{ connects } U \text{ and } U'\}$.

Mixing lemma (Alon and Chung [1988])

$\left|E(U, U')\right| = \frac{d}{|V|} \cdot |U| \cdot |U'| + \tau, \quad \text{where} \quad |\tau| \leq \lambda_2 \sqrt{|U||U'|}.$

($\bigstar\bigstar$): if $|U| = \alpha|V|$, the average degree in $G|_U$ is $\frac{|E(U,U)|}{|U|} \approx \alpha \cdot d$.

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = \textit{red} \mid a_v = \textit{red}) \geq p$

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = \textit{red} \mid a_v = \textit{red}) \geq p$

Question

Is this true for groups? Namely,

$\mathbb{P}\big(\theta = \textit{red} \mid (a_v)_{v \in U} = \textit{red}\big) \geq p$ for any $U \subset V$?

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = \mathit{red} \mid a_v = \mathit{red}) \geq p$

Question

Is this true for groups? Namely,

$\mathbb{P}\big(\theta = \mathit{red} \mid (a_v)_{v \in U} = \mathit{red}\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Open problem: puzzling unanimity
back to summary

For any agent v, $\mathbb{P}(\theta = red \mid a_v = red) \geq p$

Question

Is this true for groups? Namely,
$\mathbb{P}\big(\theta = red \mid (a_v)_{v \in U} = red\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Difficulty:
non-monotonicity of the posterior: more *red* actions observed may signal about herding $\Rightarrow$ weaker evidence.

Open problem: puzzling unanimity back to summary

For any agent v, $\mathbb{P}(\theta = \textit{red} \mid a_v = \textit{red}) \geq p$

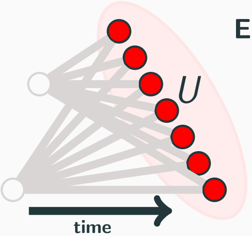
Question

Is this true for groups? Namely,
$\mathbb{P}\big(\theta = \textit{red} \mid (a_v)_{v \in U} = \textit{red}\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Difficulty:

non-monotonicity of the posterior: more *red* actions observed may signal about herding $\Rightarrow$ weaker evidence.



Example with fixed arrival order

- Strong evidence for $\theta = \textit{red}$?
- What if one observation was *blue*?

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = red \mid a_v = red) \geq p$

Question

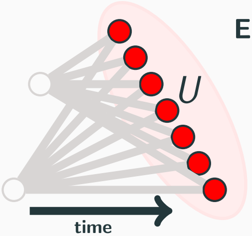
Is this true for groups? Namely,

$\mathbb{P}\big(\theta = red \mid (a_v)_{v \in U} = red\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Difficulty:

non-monotonicity of the posterior: more *red* actions observed may signal about herding $\Rightarrow$ weaker evidence.



Example with fixed arrival order

- Strong evidence for $\theta = red$? **NO**
- What if one observation was *blue*?

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = \textit{red} \mid a_v = \textit{red}) \geq p$

Question

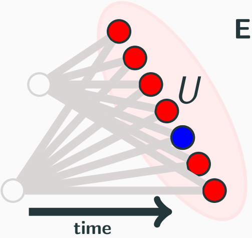
Is this true for groups? Namely,

$\mathbb{P}\big(\theta = \textit{red} \mid (a_v)_{v \in U} = \textit{red}\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Difficulty:

non-monotonicity of the posterior: more *red* actions observed may signal about herding $\Rightarrow$ weaker evidence.



Example with fixed arrival order

- Strong evidence for $\theta = \textit{red}$? **NO**
- What if one observation was *blue*?

Open problem: puzzling unanimity

back to summary

For any agent v, $\mathbb{P}(\theta = red \mid a_v = red) \geq p$

Question

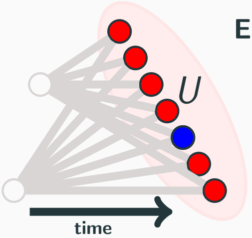
Is this true for groups? Namely,

$\mathbb{P}\big(\theta = red \mid (a_v)_{v \in U} = red\big) \geq p$ for any $U \subset V$?

Remark: if yes, an agent observing U, will (weakly) prefer the unanimous decision to his own signal $\Rightarrow$ *red* propagates.

Difficulty:

non-monotonicity of the posterior: more *red* actions observed may signal about herding $\Rightarrow$ weaker evidence.



Example with fixed arrival order

- Strong evidence for $\theta = red$? **NO**
- What if one observation was *blue*? then **YES**

Robustness to random-subset elimination

Proposition

Arbitrary $G = (V, E)$ with learning quality $L(G) = 1 - \delta$.
For uniformly random $U \subset V$ such that $|U| = \alpha \cdot |V|$, the subnetwork satisfies

$$\mathbb{E}\big[L\big(G|_U\big)\big] \geq 1 - \frac{\delta}{\alpha}.$$

Robustness to random-subset elimination

Proposition

Arbitrary $G = (V, E)$ with learning quality $L(G) = 1 - \delta$.
For uniformly random $U \subset V$ such that $|U| = \alpha \cdot |V|$, the subnetwork satisfies

$$\mathbb{E}\big[L\big(G|_U\big)\big] \geq 1 - \frac{\delta}{\alpha}.$$

Proof sketch

- Coupling between learning on G and the choice of U:

$$U = \{\text{the set of } \alpha \cdot |V| \ \text{ earliest arrivals}\}.$$

- Learning on $G|_U$ becomes a part of learning on $G \Rightarrow$

$$L(G) \leq \alpha \cdot \mathbb{E}\big[L\big(G|_U\big)\big] + (1 - \alpha) \cdot 1. \qquad \square$$

Robustness to random-subset elimination

Proposition

Arbitrary $G = (V, E)$ with learning quality $L(G) = 1 - \delta$.
For uniformly random $U \subset V$ such that $|U| = \alpha \cdot |V|$, the subnetwork satisfies

$$\mathbb{E}\big[L\big(G|_U\big)\big] \geq 1 - \frac{\delta}{\alpha}.$$

Proof sketch

- Coupling between learning on G and the choice of U:

$$U = \{\text{the set of } \alpha \cdot |V| \text{ earliest arrivals}\}.$$

- Learning on $G|_U$ becomes a part of learning on $G \Rightarrow$

$$L(G) \leq \alpha \cdot \mathbb{E}\big[L\big(G|_U\big)\big] + (1-\alpha) \cdot 1. \qquad \square$$

Robustness to random-subset elimination

Proposition

Arbitrary $G = (V, E)$ with learning quality $L(G) = 1 - \delta$.
For uniformly random $U \subset V$ such that $|U| = \alpha \cdot |V|$, the subnetwork satisfies

$$\mathbb{E}\big[L\big(G|_U\big)\big] \geq 1 - \frac{\delta}{\alpha}.$$

Proof sketch

- Coupling between learning on G and the choice of U:

$$U = \{\text{the set of } \alpha \cdot |V| \text{ earliest arrivals}\}.$$

- Learning on $G|_U$ becomes a part of learning on $G \Rightarrow$

$$L(G) \leq \alpha \cdot \mathbb{E}\big[L\big(G|_U\big)\big] + (1 - \alpha) \cdot 1. \qquad \square$$

Example: celebrity graph, $\alpha = 50\% \Rightarrow \simeq 50\%$ celebrities remain.