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ALGORITHMS FOR COMPETITIVE DIVISION OF CHORES

PROBLEM OF FAIR DIVISION

- **n agents** with **different tastes** over **m resources**
- **The goal:** find **«Fair»** and **Pareto optimal allocation**, no money transfers

  - **Applications:** dissolving partnership (e.g., divorce), seats at over-demanded courses, CPU and RAM in a cloud, charity

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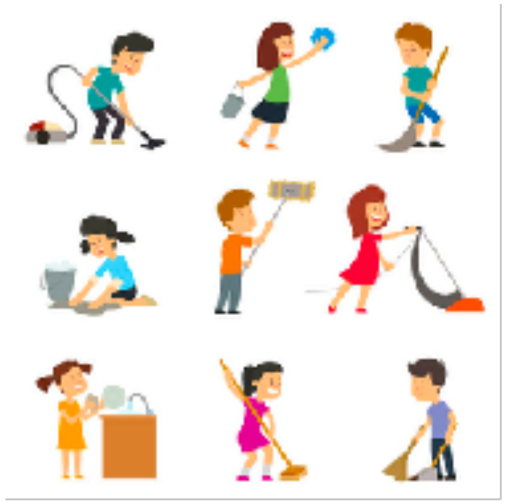

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Goods / bads problems are surprisingly different!

[Peterson, Su. (2002, 2009)], [Bogomolnaia, Moulin, Sandomirskiy, Yanovskaya (2017,2018)], [Segal-Halevi 2017]

PLAN FOR TODAY

- **Known results: divisible items (goods or bads), additive utilities**
    - Competitive Rule* = best mechanism for additive agents
        - goods: a convex optimization problem (Eisenberg- Gale)
        - bads: non-convexity, multiplicity
- **Computing all competitive allocations of bads in polynomial time for fixed n or m**
    - Enumerating demand structures of all Pareto optimal allocations
    - Finding competitive allocation with given demand structure
- **Extensions: indivisibile bads, constrained economies**

*aka Competitive Equilibrium with Equal Incomes (CEEI), Virtual Market Mechanism, Fisher Market equilibrium, or equilibrium of Arrow-Debreu exchange economy

KNOWN RESULTS

THE MODEL

- n agents, m divisible items*, $v_{i,j}$ is the value of agent i for item j
- goods: $v_{i,j} > 0$ bads: $v_{i,j} < 0$
- utility of agent i for a bundle $x = (x_1, x_2, ..x_m) \in \mathbb{R}_+^m$

$$V_i(x) = \sum_{j \in [m]} v_{i,j} x_j$$

- allocation z is a collection of bundles $(z_i)_{i \in [n]}$ with the condition

$$\sum_{i \in [n]} z_{i,j} = 1 \ \forall j \in [m]$$

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DESIRED PROPERTIES

Fairness (envy-freeness): $V_i(z_i) \geq V_i(z_k) \ \forall i, k \in [n]$

Efficiency (Pareto optimality): there is no allocation y such that $V_i(y_i) \geq V_i(z_i) \ \forall i$ and $\exists i \ V_i(y_i) > V_i(z_i)$.

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COMPETITIVE ALLOCATIONS

- **Equal choice opportunities lead to fairness:** Alice and Bob love different candies. Alice has 100$ and Bob has 100$. Both go to a supermarket and spend their money. Do they envy each other?

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DEFINITION

An allocation z is **competitive** if there **exists** a **vector of prices** $p \in \mathbb{R}^m_-$ such that for any agent i his bundle z_i **maximizes** $V_i(z_i)$ **on the budget constraint** $\langle p, z_i \rangle \leq -1$

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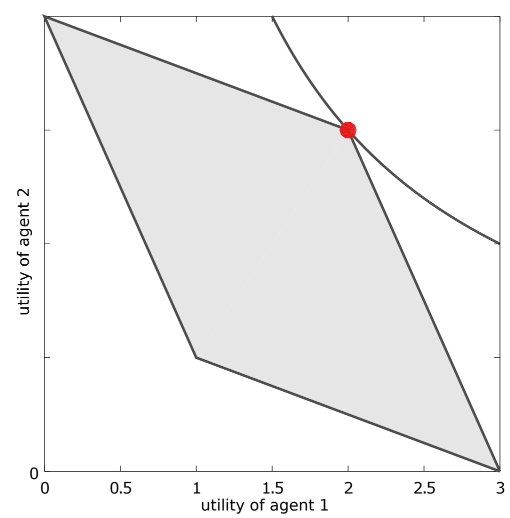
PROPERTIES OF COMPETITIVE ALLOCATIONS

- Existence, envy-freeness, Pareto optimality (the First Welfare Theorem)
- Link to Nash Social Welfare $N(z) = \prod_{i \in n} |V_i(z_i)|$

GOODS

Competitive allocation is the **global maximum of NSW**

[Eisenberg Gale (1959)]



- Convex problem => uniqness (in the space of utilities)

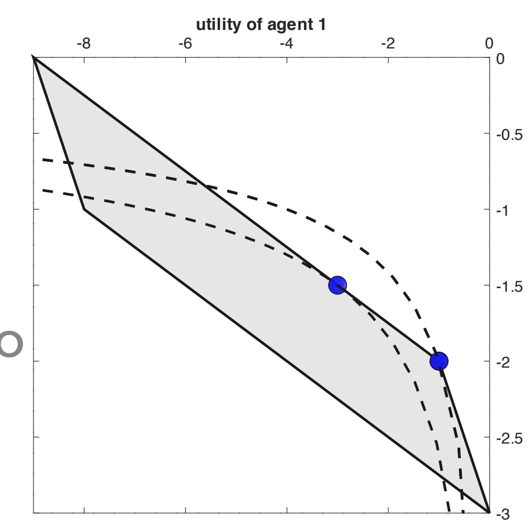
ALGORITHMS

- **approximate** by gradient decent
- **exact** by primal dual-schema
  - [Devanur, Papadimitriou, Saberi, Vazirani 2002],
  - [Orlin 2010], polynomial in n+m

NSW is used as a potential to ensure finiteness of price-adjustment procedure. **Relies on convexity!**

BADS

Competitive allocations are critical points of NSW (**local minima, local maxima, saddle points**) on the Pareto frontier. Global extrema are not competitive.



[Bogomolnaia, Moulin, Sandomirskiy, Yanovskaya (2017)]

- Non-convex problem => many allocations with different utility profiles

ALGORITHMS

NEW RESULTS: COMPUTING COMPETITIVE ALLOCATIONS OF BADS

THE MAIN RESULT

For **fixed n or m**

- **all competitive utility profiles**
- a set of **competitive allocations, one per utility profile**

can be computed in **strongly polynomial time*** as a function of matrix of values v.

*The number of elementary operations (addition, multiplication etc) is bounded by a polynomial of the free parameter (n or m); the memory used is bounded by polynomial of the input length.

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  - there are examples with $2^{\min(n,m)}$ competitive utility profiles

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  [Bogomolnaia, Moulin, Sandomirskiy, Yanovskaya (2017)]
- The algorithm gives an upper bound for the **number of competitive profiles**

$$\min\left\{(2m+1)^{\frac{n(n-1)}{2}}, (2n+1)^{\frac{m(m-1)}{2}}\right\}$$

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IDEAS

Consumption graph $G(z)$: bipartite graph on (agents–bads), where i and j are connected if $z_{i,j} > 0$

OBSERVATION

Finding a competitive allocation (if exists) for a given consumption graph G is easy*.

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- Fixing G = fixing a face of the Pareto frontier
- For a given face, FOCs of criticality of NSW give exact formula for $V = (V_i(z_i))_{i \in [n]}$ if there is a competitive allocation z with $G(z) = G$
- For a given vector V, existence of competitive z can be checked using the auxiliary MaxFlow problem of [Devanur, Papadimitriou, Saberi, Vazirani 2002]

THE ALGORITHM

```
for G ∈ the set of all (n,m)-bipartite graphs {
    compute a competitive allocation z with G(z) = G
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- **Example:** the set of all graphs, the set of all efficient consumption graphs

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find rich polynomial set of graphs

FINDING A RICH SET

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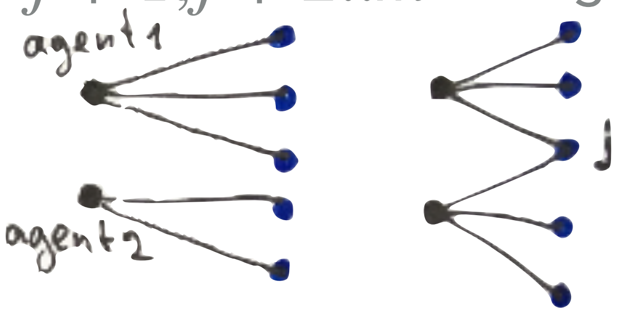
- **n=2:** any efficient allocation has the following structure:
  - reorder bads by decreasing of $\frac{|v_{2,j}|}{|v_{1,j}|}$. Fix a bad $j \in [m]$,
  - give $1,2..,j-1$ to agent 1, $j+1,j+2..m$ to agent 2 and split j arbitrarily
  - 2m+1 consumption graphs



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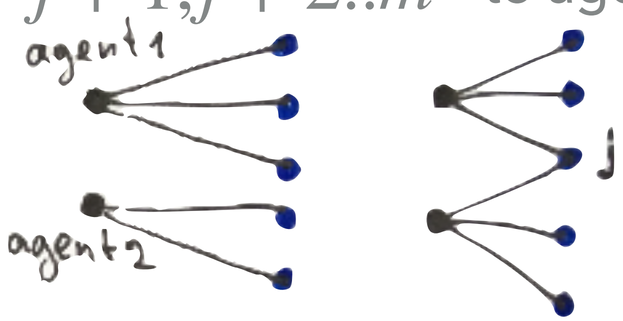

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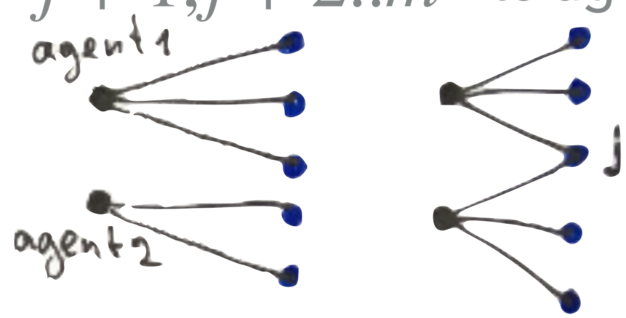

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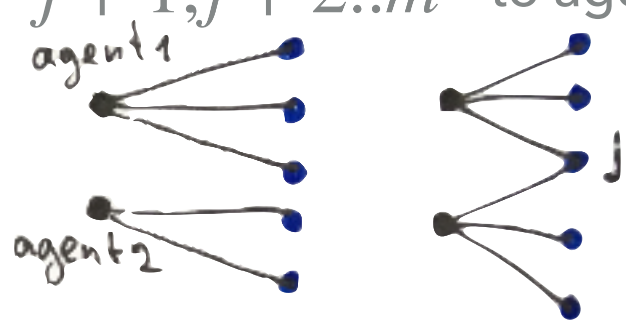

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- **fixed m, large n:** use the duality (corollary of the 2nd Welfare Th):

EFFG is invariant w.r.t. to changing the roles of agents and items

EXTENSIONS

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For a given «divisible» competitive allocation z , there is a competitive allocation z' with **unequal budgets** such that:

- z' **is integral** (no items are shared).
- **budgets are close** $\left| \, |b_i'| - 1 \right| \leq \max_{j \in [m]} |p_j|$ for all agents i

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$$V_i(z_i \setminus \{j\}) \geq V_i(z_k \cup \{j'\}) \text{ for some } j, j' \in [m]$$

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First result on existence of approx fair allocation o bads

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COROLLARY

For fixed n or m, a **Pareto-Optimal Envy–Free-(1,1)** allocation of **indivisible bads** can be computed in **strongly polynomial time.**

CONSTRAINED ECONOMIES (OPEN PROBLEM)

economy with bads <=> constrained economy with goods:

[Bogomolnaia, Moulin, Sandomirskiy, Yanovskaya (2017)]

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- mixture of goods and bads
- assignment problems [Hylland, Zeckhauser 1979]: $\sum_{j \in [m]} z_{ij} = \frac{m}{n}$
  - Complicated algorithm: [Alaei, Khalilabadi, Tardos 2017]
- Upper and lower bounds on consumption of a subset of items

COMPUTING ONE COMPETITIVE ALLOCATION (OPEN PROBLEM)

If **n and m** are both **large**, **no hope to compute ALL** competitive **allocations** (may have exponential number of them even in the utility space)

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Thank you! (open) questions? (closing) remarks?

BIBLIOGRAPHY

- S. Branzei, F. Sandomirskiy Algorithms for Competitive Division of Chores. To appear on arXiv and at EC 2019 (hopefully)
- E. Peterson, F. Su. 2009. N-person envy-free chore division. arXiv:0909.0303
- Erel Segal-Halevi Fairly Dividing a Cake after Some Parts Were Burnt in the Oven arXiv:1704.00726 [math.CO]
- Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. Com- petitive division of a mixed manna. Econometrica, 85:1847–1871, 2017.
- H. Varian. 1974. Equity, envy and efficiency. Journal of Economic Theory 9, 63–91.
- E. Eisenberg. 1961. Aggregation of utility functions, Management Science, 7, 337-350.
- Siddharth Barman, Sanath Kumar Krishnamurthy On the Proximity of Markets with Integral Equilibria, arXiv:1811.08673 [cs.GT]
- Saeed Alaei, Pooya Jalaly Khalilabadi, and Eva Tardos. Computing equilibrium in matching markets. In Proceedings of the 2017 ACM Conference on Economics and Computation, EC ’17, pages 245–261, New York, NY, USA, 2017. ACM.
- Aanund Hylland and Richard Zeckhauser. The efficient allocation of individuals to positions. Journal of Political economy, 87(2):293–314, 1979.