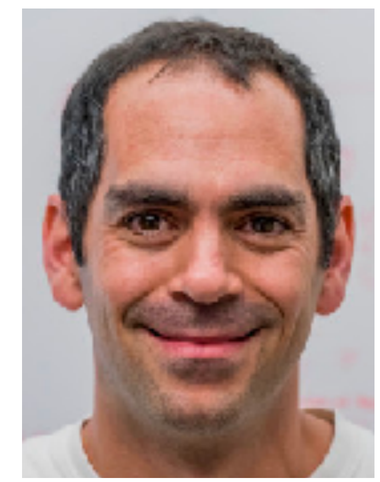


ITAI ARIELI (TECHNION)
YAKOV BABICHENKO (TECHNION)
FEDOR SANDOMIRSKIY (TECHNION, HSE ST.PETERSBURG)
OMER TAMUZ (CALTECH)

FEASIBLE JOINT POSTERIOR BELIEFS

arXiv:2002.11362

BAYESIAN COMMUNICATION

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

Random state:

$$\theta = \begin{cases} \textbf{1, } \text{prob.} & p \\ \textbf{0, } \text{prob.} & 1-p \end{cases}$$

$s_1 \in S_1$

$\vdots$

$s_N \in S_N$

$\vdots$

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

signals with joint distribution

$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$

BAYESIAN COMMUNICATION

Random state:

$$\theta = \begin{cases} \textbf{1}, \text{ prob.} & p \\ \textbf{0}, \text{ prob.} & 1-p \end{cases}$$

$s_1 \in S_1$

⋮

$s_N \in S_N$

signals with joint distribution

$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

⋮

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

?

What joint distributions of posteriors on $[0,1]^N$ are feasible*?

*can be induced by some signalling policy

$\left(S_1 \ldots S_N,\ \mathbb{P}(s_1 \ldots s_N \mid \theta)\right)$

BAYESIAN COMMUNICATION

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

$s_1 \in S_1$

$\vdots$

$s_N \in S_N$

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

Random state:

$$\theta = \begin{cases} \textbf{1}, \text{ prob. } & p \\ \textbf{0}, \text{ prob. } & 1 - p \end{cases}$$

signals with joint distribution

$$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$$

WHY IMPORTANT?

? **What joint distributions of posteriors on $[0,1]^N$ are feasible*?**

*can be induced by some signalling policy

$$\left(S_1 \ldots S_N,\ \mathbb{P}(s_1 \ldots s_N \mid \theta)\right)$$

BAYESIAN COMMUNICATION

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

Random state:

$$\theta = \begin{cases} \textbf{1}, \text{ prob. } & p \\ \textbf{0}, \text{ prob. } & 1-p \end{cases}$$

$s_1 \in S_1$

⋮

$s_N \in S_N$

⋮

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

signals with joint distribution

$$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$$

?

What joint distributions of posteriors on $[0,1]^N$ are feasible*?

*can be induced by some signalling policy

$$\left(S_1 \ldots S_N, \ \mathbb{P}(s_1 \ldots s_N \mid \theta)\right)$$

WHY IMPORTANT?

- **Learning theory:** which distributions can be the outcome of a learning process?

BAYESIAN COMMUNICATION

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

$s_1 \in S_1$

Random state:

$$\theta = \begin{cases} \textbf{1}, \text{ prob. } & p \\ \textbf{0}, \text{ prob. } & 1-p \end{cases}$$

$s_N \in S_N$

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

signals with joint distribution

$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$

?

What joint distributions of posteriors on $[0,1]^N$ are feasible*?

*can be induced by some signalling policy

$\left(S_1 \ldots S_N,\ \mathbb{P}(s_1 \ldots s_N \mid \theta)\right)$

WHY IMPORTANT?

- **Learning theory:** which distributions can be the outcome of a learning process?
- **Decision theory:** which distributions can be rationalised within the Bayesian framework?

BAYESIAN COMMUNICATION

Random state:

$$\theta = \begin{cases} \textbf{1}\text{, prob.} & p \\ \textbf{0}\text{, prob.} & 1-p \end{cases}$$

$s_1 \in S_1$

$\vdots$

$s_N \in S_N$

signals with joint distribution

$$\mathbb{P}(s_1, s_2 \ldots, s_N \mid \theta)$$

N Receivers:

POSTERIOR

$p_1 = \mathbb{P}(\theta = 1 \mid s_1)$

$\vdots$

POSTERIOR

$p_N = \mathbb{P}(\theta = 1 \mid s_N)$

?

What joint distributions of posteriors on $[0,1]^N$ are feasible*?

*can be induced by some signalling policy

$$\big(S_1 \ldots S_N,\ \mathbb{P}(s_1 \ldots s_N \mid \theta)\big)$$

WHY IMPORTANT?

- **Learning theory:** which distributions can be the outcome of a learning process?
- **Decision theory:** which distributions can be rationalised within the Bayesian framework?
- **Bayesian persuasion:** which distributions can be induced by the designer?

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

NO ANALOG OF SPLITTING LEMMA WAS KNOWN

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$

SEVERAL RECEIVERS:

~~NO ANALOG OF SPLITTING LEMMA WAS KNOWN~~

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

~~NO ANALOG OF SPLITTING LEMMA WAS KNOWN~~

- **Dawid, DeGroot, Mortera (1995)**
  - Mysterious characterisation for N=2

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

~~NO ANALOG OF SPLITTING LEMMA WAS KNOWN~~

- **Dawid, DeGroot, Mortera (1995)**
  - Mysterious characterisation for N=2
- **Burdzy, Pitman (2020)**
  - Bounds on how different the beliefs can be for N=2

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\displaystyle\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

~~NO ANALOG OF SPLITTING LEMMA WAS KNOWN~~

- **Dawid, DeGroot, Mortera (1995)**
  - Mysterious characterisation for N=2
- **Burdzy, Pitman (2020)**
  - Bounds on how different the beliefs can be for N=2
- **Ziegler (2020), Levy, de Barreda, Razin (2020)**
  - Necessary conditions

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\Longleftrightarrow$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

~~NO ANALOG OF SPLITTING LEMMA WAS KNOWN~~

- **Dawid, DeGroot, Mortera (1995)**
  - Mysterious characterisation for N=2
- **Burdzy, Pitman (2020)**
  - Bounds on how different the beliefs can be for N=2
- **Ziegler (2020), Levy, de Barreda, Razin (2020)**
  - Necessary conditions
- **Morris (2020)**
  - Characterisation of feasibility <= No-trade characterisation of Common Prior

KNOWN RESULTS

N=1 RECEIVER:

The Martingale Property: $\mathbb{E}[p_1] = p$

- On average, the belief does not change

SPLITTING LEMMA (R.AUMANN & M.MASCHLER / D.BLACKWELL)

μ on $[0,1]$ is feasible

$\iff$ satisfies MP: $\int_{[0,1]} p_1 \, d\mu(p_1) = p$



SEVERAL RECEIVERS:

- **Dawid, DeGroot, Mortera (1995)**
  - Mysterious characterisation for N=2
- **Burdzy, Pitman (2020)**
  - Bounds on how different the beliefs can be for N=2
- **Ziegler (2020), Levy, de Barreda, Razin (2020)**
  - Necessary conditions
- **Morris (2020)**
  - Characterisation of feasibility <= No-trade characterisation of Common Prior

NO ANALOG OF SPLITTING LEMMA WAS KNOWN FOR N>2

PLAN FOR TODAY

- **Characterisations of feasibility**
  - **N=2:** Agreement Theorem & Theorem of Dawid et al. (1995)
    - Independent beliefs
  - **N>2:** Characterisation via no-trade
- **Bayesian Persuasion**
  - **Optimal policies** as extreme points of feasible distributions
  - **Example:** inducing a conflict via Hilbert-space geometry.

CHARACTERISATIONS OF FEASIBILITY

N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary

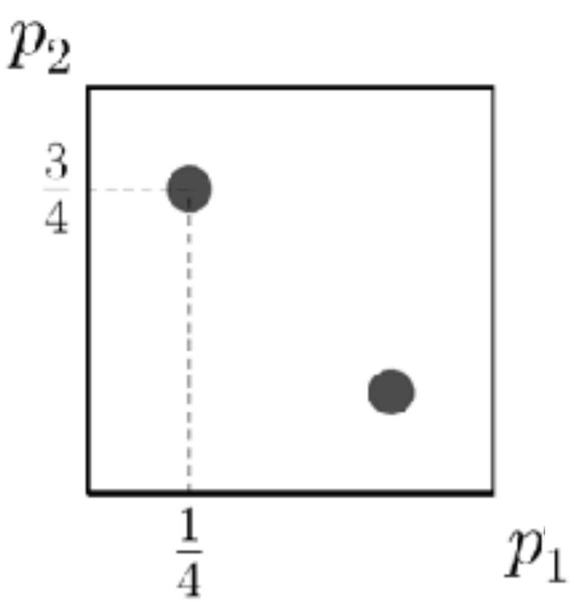
N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary
- but not sufficient

N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary
- but not sufficient

EXAMPLE WITH $p = 1/2$



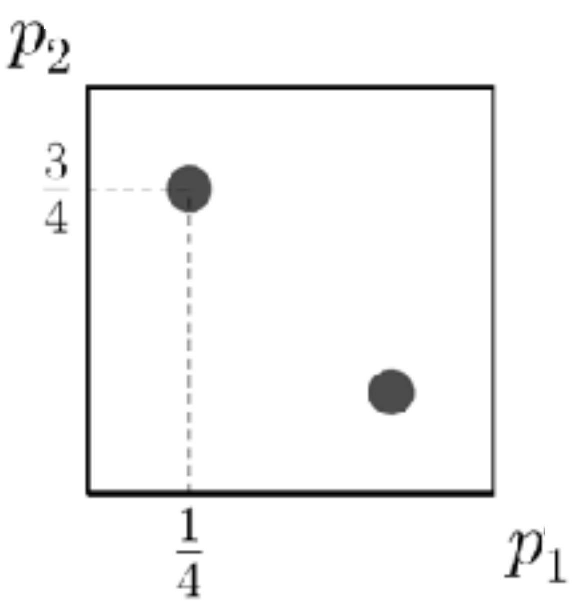
Infeasible:

- Posteriors are common knowledge
- Bayesian-rationals cannot **agree to disagree** Aumann (1976)

N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary
- but not sufficient

EXAMPLE WITH $p = 1/2$



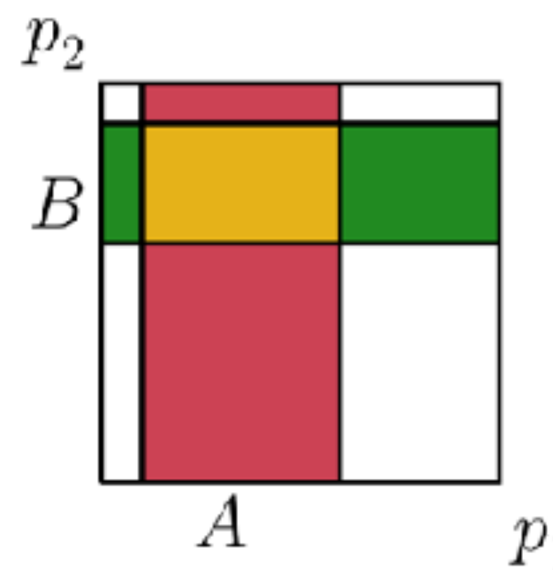
Infeasible:

- Posteriors are common knowledge
- Bayesian-rationals cannot **agree to disagree** Aumann (1976)

QUANTITATIVE BOUND ON DISAGREEMENT

- **Define** $\delta(A, B) =$

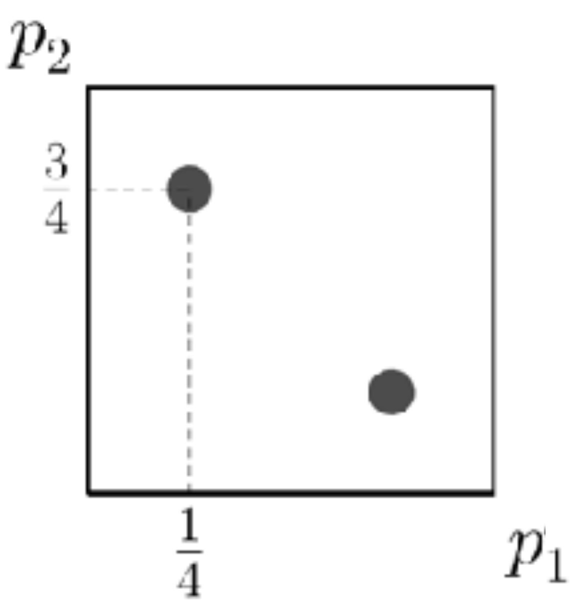
$$= \int_{A \times [0,1]} p_1 \, d\mu - \int_{[0,1] \times B} p_2 \, d\mu$$



N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary
- but not sufficient

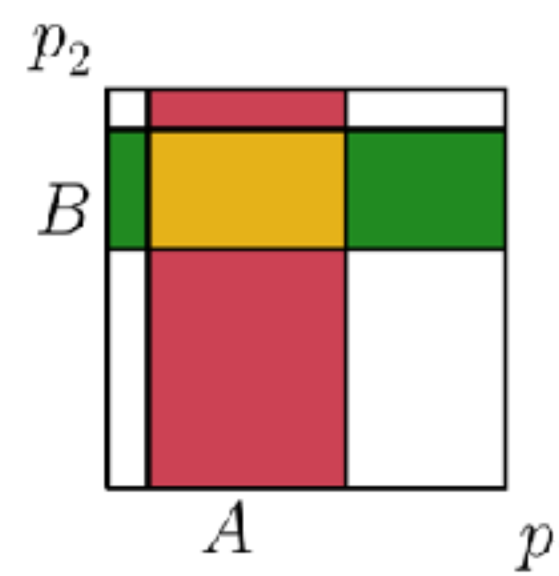
EXAMPLE WITH $p = 1/2$



Infeasible:

- Posteriors are common knowledge
- Bayesian-rationals cannot **agree to disagree** Aumann (1976)

QUANTITATIVE BOUND ON DISAGREEMENT



- **Define** $\delta(A, B) =$

$$= \int_{A \times [0,1]} p_1 \, d\mu - \int_{[0,1] \times B} p_2 \, d\mu$$

- **Then**

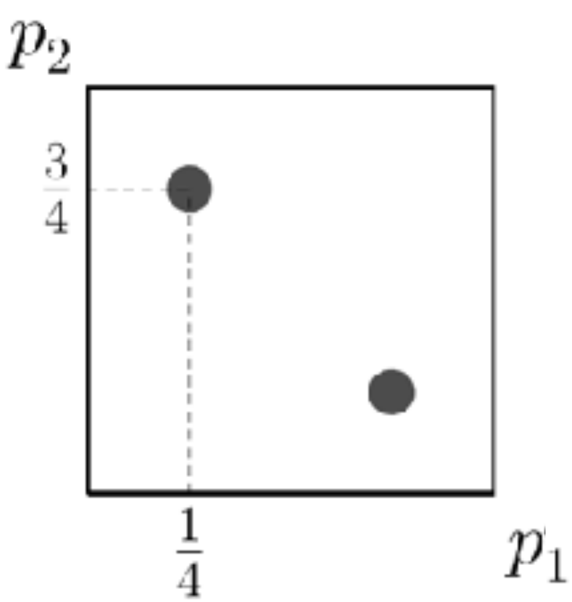
$$\mu(A \times \overline{B}) \geq \delta(A, B) \geq -\mu(\overline{A} \times B)$$

for any feasible μ **and** $A, B \subset [0,1]$.

N=2: IS MARTINGALE PROPERTY ENOUGH FOR FEASIBILITY?

- the MP $p = \int_{[0,1]^2} p_1 \, d\mu(p_1, p_2) = \int_{[0,1]^2} p_2 \, d\mu(p_1, p_2)$ is necessary
- but not sufficient

EXAMPLE WITH $p = 1/2$



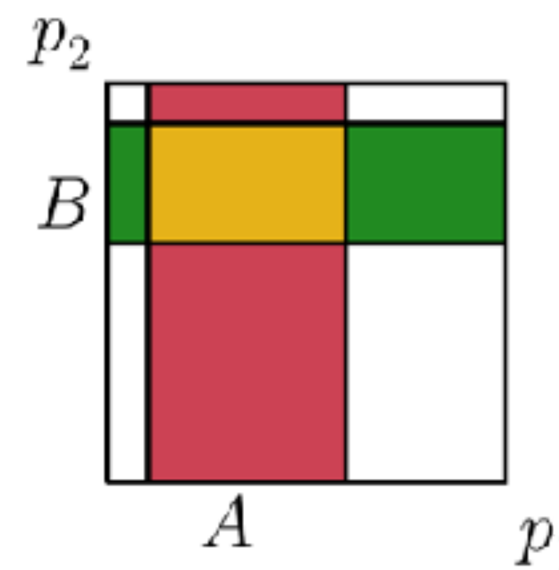
Infeasible:

- Posteriors are common knowledge
- Bayesian-rationals cannot **agree to disagree** Aumann (1976)

QUANTITATIVE BOUND ON DISAGREEMENT

- **Define** $\delta(A, B) =$

$$= \int_{A\times[0,1]} p_1 \, d\mu - \int_{[0,1]\times B} p_2 \, d\mu$$



- **Then**

$$\mu(A \times \bar{B}) \geq \delta(A, B) \geq -\mu(\bar{A} \times B)$$

for any feasible μ **and** $A, B \subset [0,1]$.

AUMANN'S AGREEMENT THEOREM

$$\iff \mu(A \times \bar{B}) = \mu(\bar{A} \times B) = 0$$

FEASIBILITY FOR N=2

THEOREM (DAWID, DEGROOT, MORTERA (1995))

A distribution is feasible $\Longleftrightarrow$ satisfies

- **Martingale Property**
- **Quantitative bound on disagreement**

$$\mu(A \times \overline{B}) \geq \int_{A\times[0,1]} p_1 \, d\mu - \int_{[0,1]\times B} p_2 \, d\mu \geq -\mu(\overline{A} \times B)$$

FEASIBILITY FOR N=2

THEOREM (DAWID, DEGROOT, MORTERA (1995))

A distribution is feasible $\Longleftrightarrow$ satisfies

- **Martingale Property**
- **Quantitative bound on disagreement**

$$\mu(A \times \overline{B}) \geq \int_{A\times[0,1]} p_1 \, d\mu - \int_{[0,1]\times B} p_2 \, d\mu \geq -\mu(\overline{A} \times B)$$

- Let's see this result in action

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

feasible?

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

feasible?

Yes!

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

 feasible?

Yes!

CRITERION OF FEASIBILITY FOR PRODUCT DISTRIBUTIONS

Measure ϕ on $[0,1]$, symmetric around $\frac{1}{2}$.

$\phi \times \phi$ is **feasible** $\Longleftrightarrow$ **2nd-order dominates the Uniform**

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

feasible?

Yes!

CRITERION OF FEASIBILITY FOR PRODUCT DISTRIBUTIONS

Measure ϕ on $[0,1]$, symmetric around $\frac{1}{2}$.

$\phi \times \phi$ is **feasible** $\Longleftrightarrow$ **2nd-order dominates the Uniform**

- Is 

feasible?

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

feasible?

Yes!

CRITERION OF FEASIBILITY FOR PRODUCT DISTRIBUTIONS

Measure ϕ on $[0,1]$, symmetric around $\frac{1}{2}$.

$\phi \times \phi$ is **feasible** $\Longleftrightarrow$ **2nd-order dominates the Uniform**

- Is 

feasible? **No!**

COROLLARY: PRODUCT DISTRIBUTIONS WITH $p = 1/2$

- Can posteriors be independent?
  - R1's belief tells her nothing about the belief of R2
- Is 

feasible?

Yes!

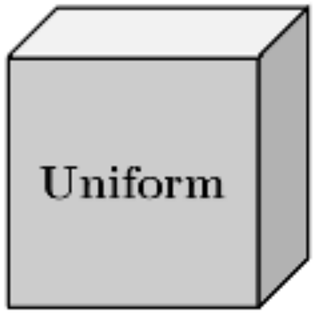
CRITERION OF FEASIBILITY FOR PRODUCT DISTRIBUTIONS

Measure ϕ on $[0,1]$, symmetric around $\frac{1}{2}$.

$\phi \times \phi$ is **feasible** $\Longleftrightarrow$ **2nd-order dominates the Uniform**

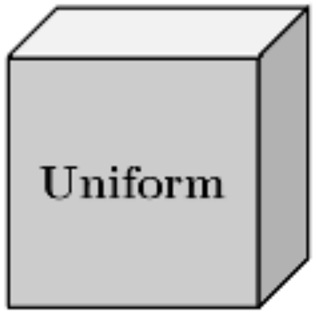
- Is 

feasible? **No!** Let's understand why



IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

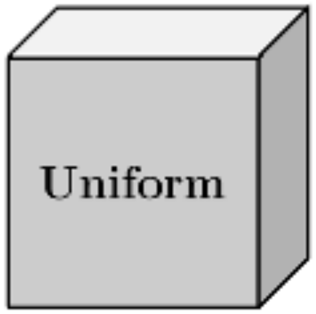


IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$



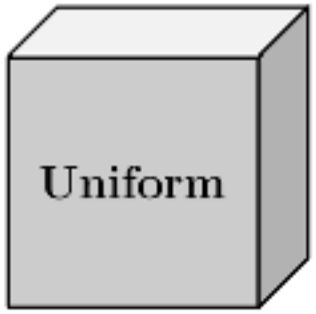
IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0



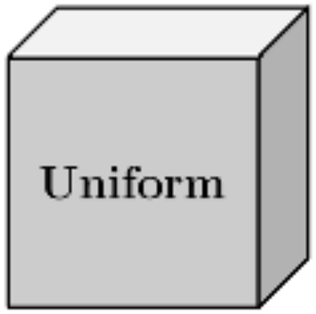
IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0
- Our profit must also be $0 = \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - 1_{\theta=1} \cdot \sum_{i=1}^{3} t(p_i)\right] \geq$



IS INFEASIBLE

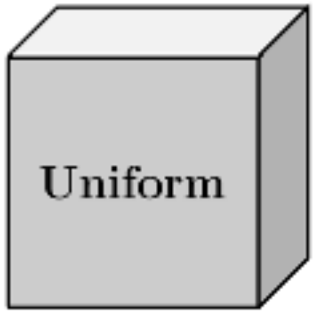
IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0
- Our profit must also be $0 = \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - 1_{\theta=1} \cdot \sum_{i=1}^{3} t(p_i)\right] \geq$

$$\geq \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] =$$



IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

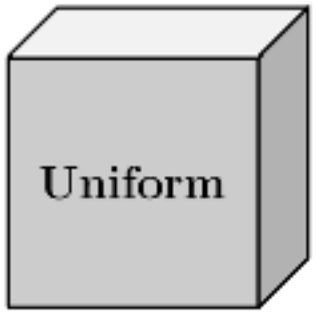
- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0
- Our profit must also be $0 = \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - 1_{\theta=1} \cdot \sum_{i=1}^{3} t(p_i)\right] \geq$

$$\geq \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] =$$

$$= \int_{[0,1]^3} \left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] dp_1 dp_2 dp_3$$



IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

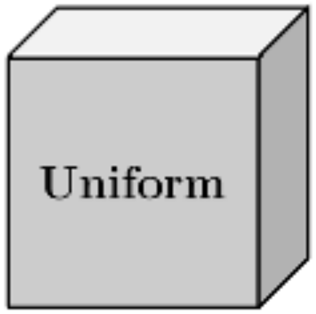
- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0
- Our profit must also be $0 = \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - 1_{\theta=1} \cdot \sum_{i=1}^{3} t(p_i)\right] \geq$

$$\geq \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] =$$

$$= \int_{[0,1]^3} \left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] dp_1 dp_2 dp_3 = \frac{1}{9} ?!?$$



IS INFEASIBLE

IF IT WAS FEASIBLE WE COULD CONSTRUCT MONEY PUMP:

- Promise to transfer $t(p_i)$ euros to **Receiver** i if $\theta = 1$:

$$t(p_i) = \begin{cases} 1, & p_i \geq 2/3 \\ -1, & p_i \leq 1/3 \end{cases}$$

- Ask to pay her fair price $\mathbb{E}[t(p_i) \cdot p_i] \Rightarrow$ her profit is 0
- Our profit must also be $0 = \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - 1_{\theta=1} \cdot \sum_{i=1}^{3} t(p_i)\right] \geq$

$$\geq \mathbb{E}\left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] =$$

$$= \int_{[0,1]^3} \left[\sum_{i=1}^{3} p_i \cdot t(p_i) - \max\left\{0, \sum_{i=1}^{3} t(p_i)\right\}\right] dp_1 dp_2 dp_3 = \frac{1}{9} ?!?$$

CONCLUSION uniform on $[0,1]^3$ is not feasible

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

THEOREM

μ is feasible for some prior $\iff$ **no money pump**

$$\int_{\Delta(\Theta)} \left[\sum_{i=1}^{N} \left(\int_{\Theta} t_i(p_i, \theta)\, dp_i \right) - \sup_{\theta \in \Theta} \left\{ \sum_{i=1}^{N} t_i(p_i, \theta) \right\} \right] d\mu(p_1, \ldots, dp_N) \leq 0$$

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

THEOREM

μ is feasible for some prior $\iff$ **no money pump**

$$\int_{\Delta(\Theta)} \left[\sum_{i=1}^{N} \left(\int_{\Theta} t_i(p_i, \theta)\, dp_i \right) - \sup_{\theta \in \Theta} \left\{ \sum_{i=1}^{N} t_i(p_i, \theta) \right\} \right] d\mu(p_1, \ldots, dp_N) \leq 0$$

PROOF

- ***Necessity*** obvious
- ***Sufficiency*** *the* Farkas lemma (finite support),

  Kellerer's theorem (1984) (general case)

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

THEOREM

μ is feasible for some prior $\iff$ **no money pump**

$$\int_{\Delta(\Theta)} \left[\sum_{i=1}^{N} \left(\int_{\Theta} t_i(p_i, \theta)\, dp_i \right) - \sup_{\theta \in \Theta} \left\{ \sum_{i=1}^{N} t_i(p_i, \theta) \right\} \right] d\mu(p_1, \ldots, dp_N) \leq 0$$

PROOF

- ***Necessity*** obvious
- ***Sufficiency*** *the* Farkas lemma (finite support),
  Kellerer's theorem (1984) (general case)

REMARK

- Theorem of Dawid et al. $\iff$ binary Θ, N=2, t_i = indicators

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

THEOREM

μ is feasible for some prior $\Longleftrightarrow$ **no money pump**

$$\int_{\Delta(\Theta)} \left[\sum_{i=1}^{N} \left(\int_{\Theta} t_i(p_i, \theta)\, dp_i \right) - \sup_{\theta \in \Theta} \left\{ \sum_{i=1}^{N} t_i(p_i, \theta) \right\} \right] d\mu(p_1, \ldots, dp_N) \leq 0$$

PROOF

- ***Necessity*** obvious
- ***Sufficiency*** *the* Farkas lemma (finite support),
  Kellerer's theorem (1984) (general case)

REMARK

- Theorem of Dawid et al. $\Longleftrightarrow$ binary Θ, N=2, t_i = indicators
- For N>2, indicators are not enough

FEASIBILITY FOR N>2 AND ARBITRARY SET OF STATES

- Set of states Θ can be finite vs. countable vs. continuous
- Prior and posteriors $p, p_i \in \Delta(\Theta)$
- Joint distribution of posteriors $\mu \in \Delta\Big(\Delta(\Theta) \times \ldots \times \Delta(\Theta)\Big)$
- Transfers $t_i(p_i, \theta)$

THEOREM

μ is feasible for some prior $\Longleftrightarrow$ **no money pump**

$$\int_{\Delta(\Theta)} \left[\sum_{i=1}^{N} \left(\int_{\Theta} t_i(p_i, \theta)\, dp_i \right) - \sup_{\theta \in \Theta} \left\{ \sum_{i=1}^{N} t_i(p_i, \theta) \right\} \right] d\mu(p_1, \ldots, dp_N) \leq 0$$

PROOF

- ***Necessity*** obvious
- ***Sufficiency*** *the* Farkas lemma (finite support),
  Kellerer's theorem (1984) (general case)

REMARK

- Theorem of Dawid et al. $\Longleftrightarrow$ binary Θ, N=2, t_i = indicators
- For N>2, indicators are not enough

QUESTION

- What is enough? Say, are combinations of N-1 indicators enough?

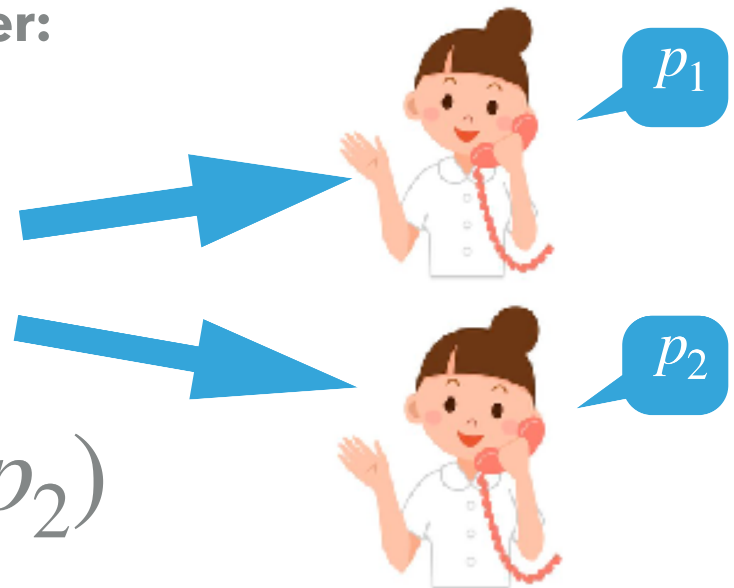
BAYESIAN PERSUASION

BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

utility $u(p_1, p_2)$

Receivers:



THE GOAL

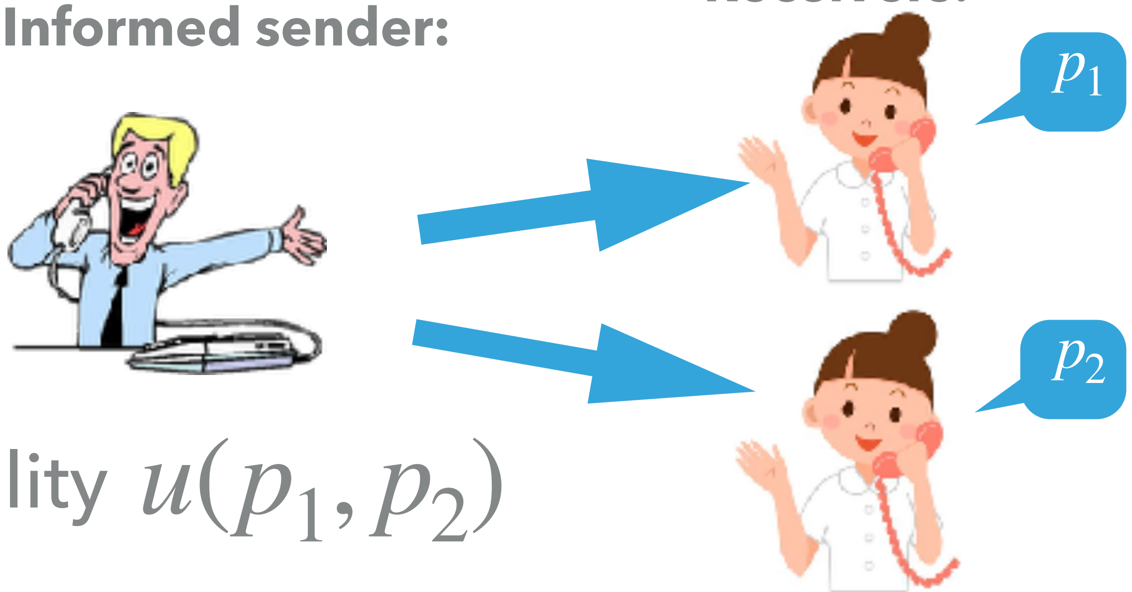
Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

Receivers:



utility $u(p_1, p_2)$

THE GOAL

Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

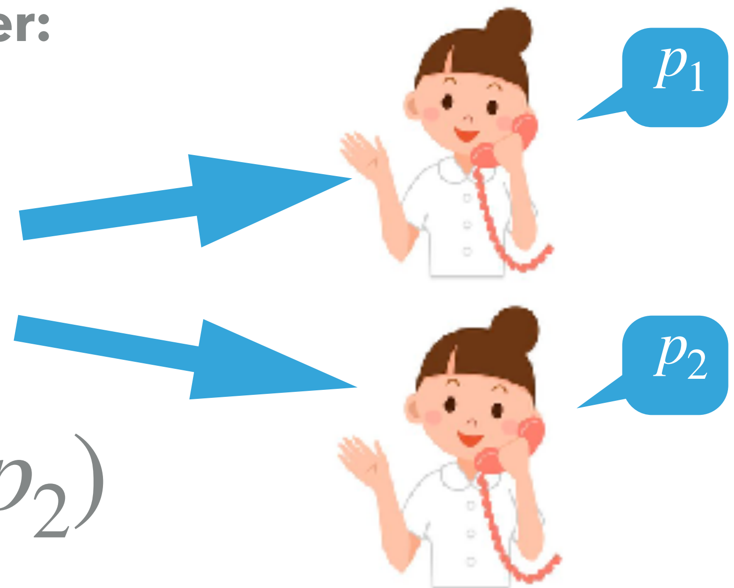
- Optimal policies = extreme points of feasible distributions

BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

utility $u(p_1, p_2)$

Receivers:

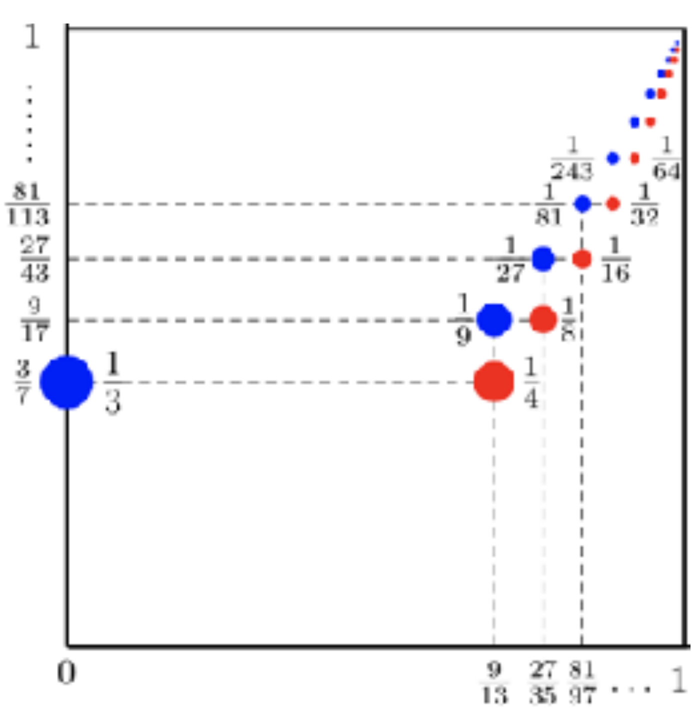


THE GOAL

Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

- Optimal policies = extreme points of feasible distributions
- May have countable support $\Longrightarrow$ infinite number of signals

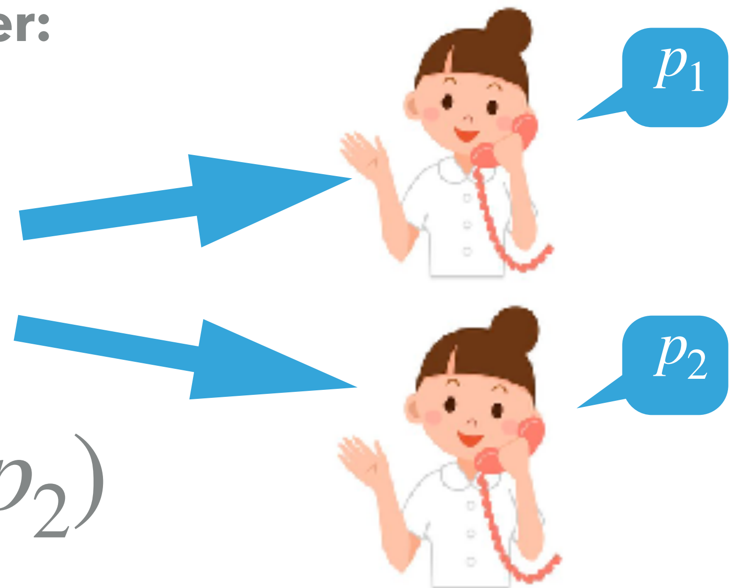


BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

utility $u(p_1, p_2)$

Receivers:

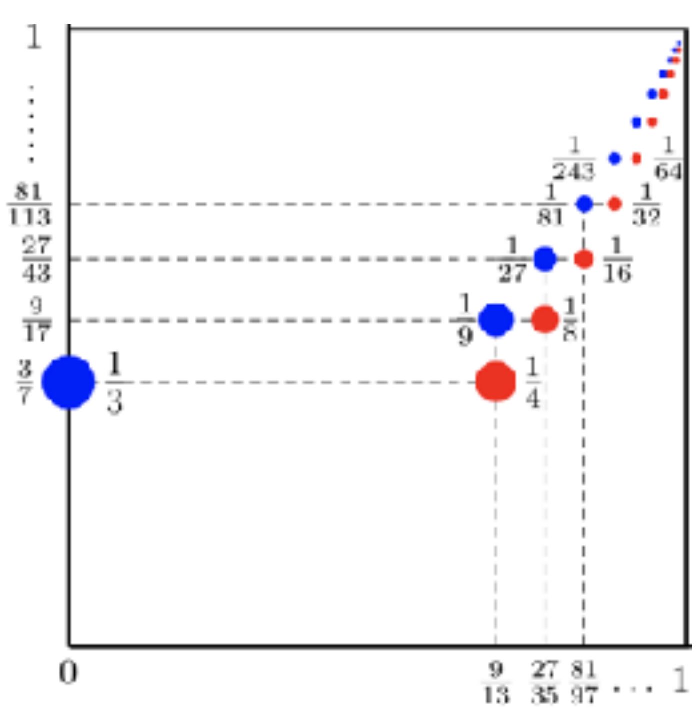


THE GOAL

Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

- Optimal policies = extreme points of feasible distributions
- May have countable support $\Longrightarrow$ infinite number of signals

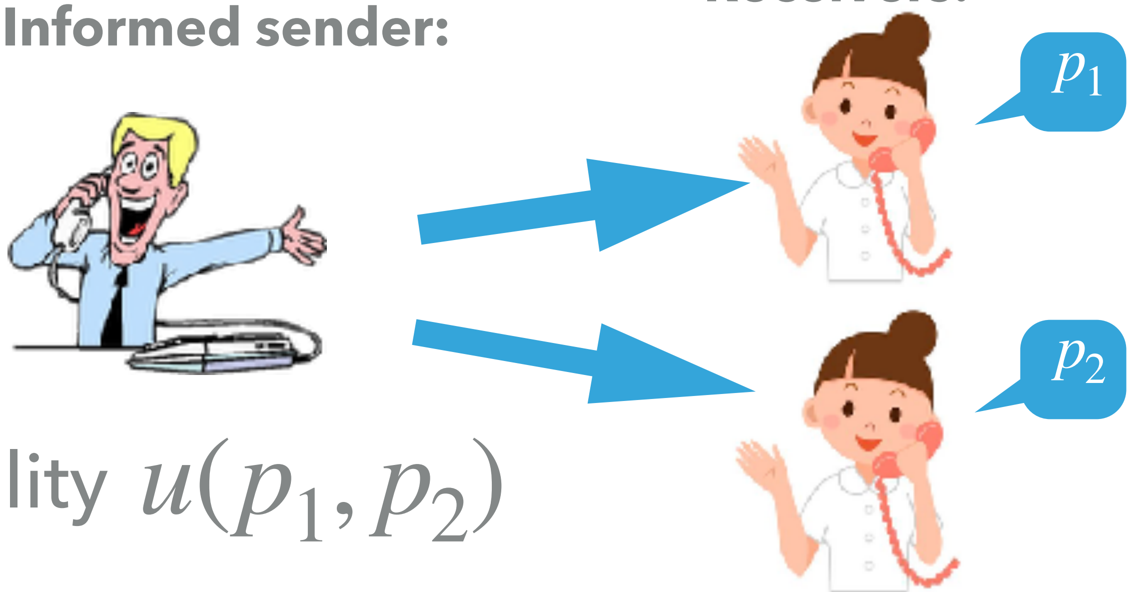


- Contrast with N=1, where 2 signals are enough
  - Kamenica, Gentzkow (2011)

BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

Receivers:



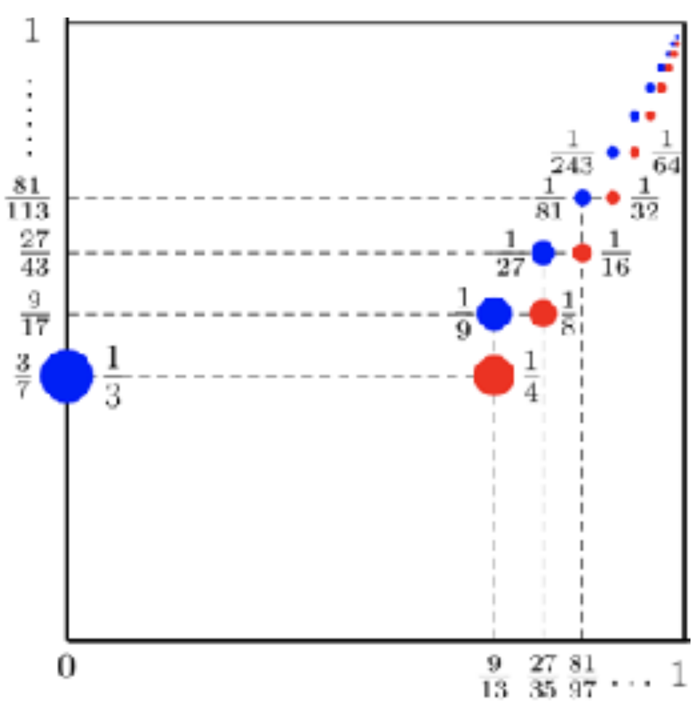
utility $u(p_1, p_2)$

THE GOAL

Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

- Optimal policies = extreme points of feasible distributions
- May have countable support $\Longrightarrow$ infinite number of signals



- Contrast with N=1, where 2 signals are enough
  - Kamenica, Gentzkow (2011)

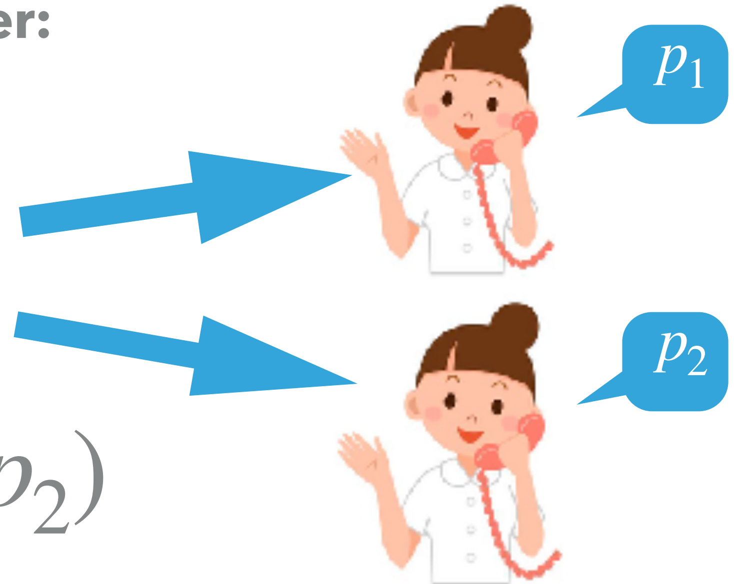
CONJECTURE No non-atomic extreme points

BAYESIAN PERSUASION WITH N=2 RECEIVERS

Informed sender:

utility $u(p_1, p_2)$

Receivers:

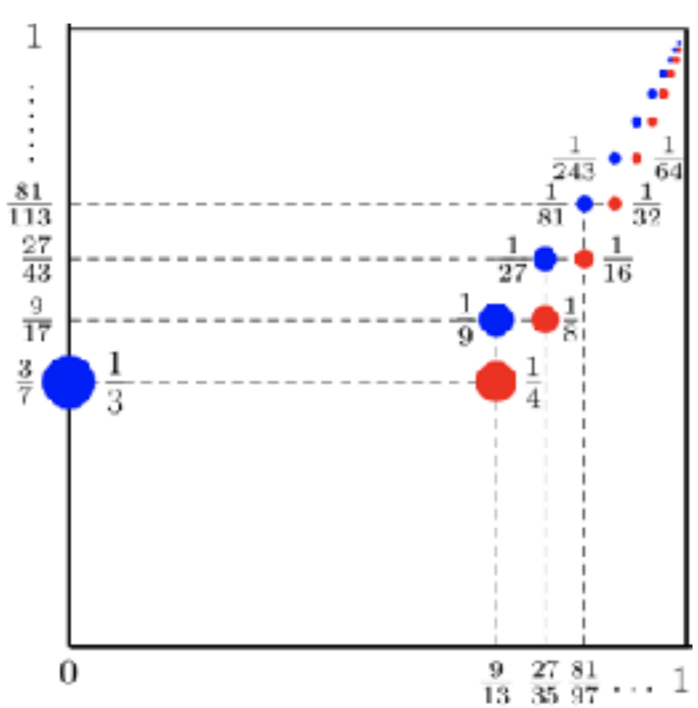


THE GOAL

Maximize $\mathbb{E}\left[u(p_1, p_2)\right]$

over feasible distributions

- Optimal policies = extreme points of feasible distributions
- May have countable support $\Longrightarrow$ infinite number of signals



- Contrast with N=1, where 2 signals are enough
  - Kamenica, Gentzkow (2011)

CONJECTURE No non-atomic extreme points

- **We proved:** any extreme point has 0-measure support

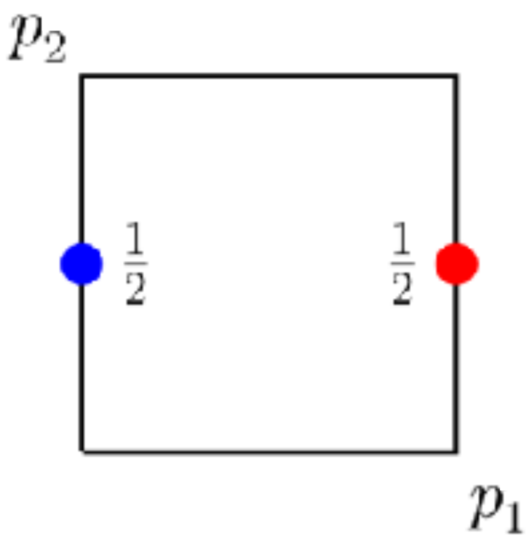
EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$

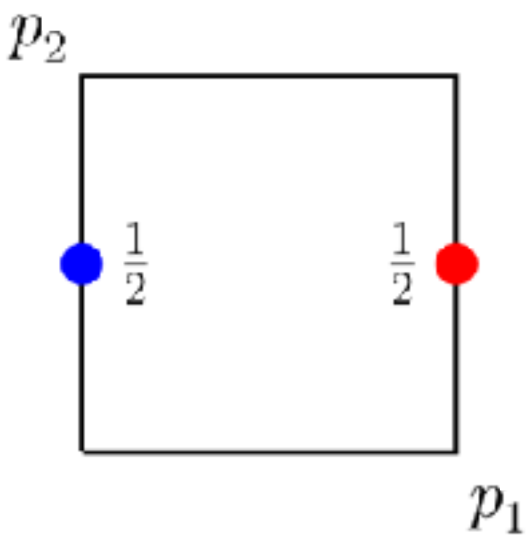
EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

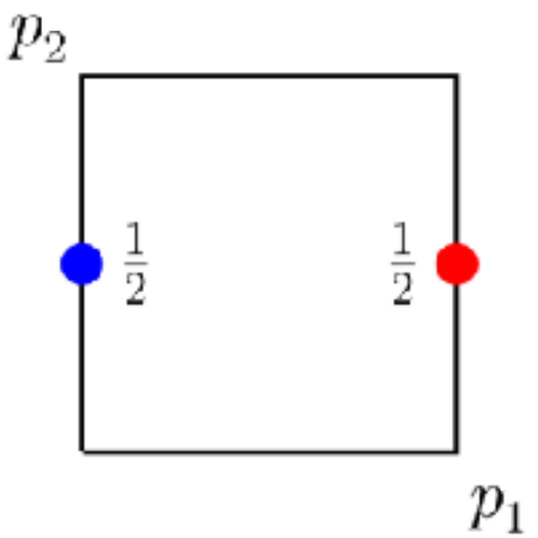
Maximize $\mathbb{E}\,|p_1 - p_2|^2$



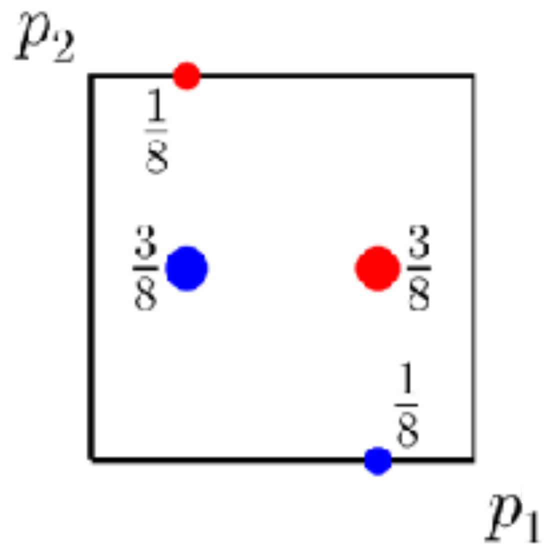
Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\left[(p_1 - 0.5)(p_2 - 0.5)\right]$

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



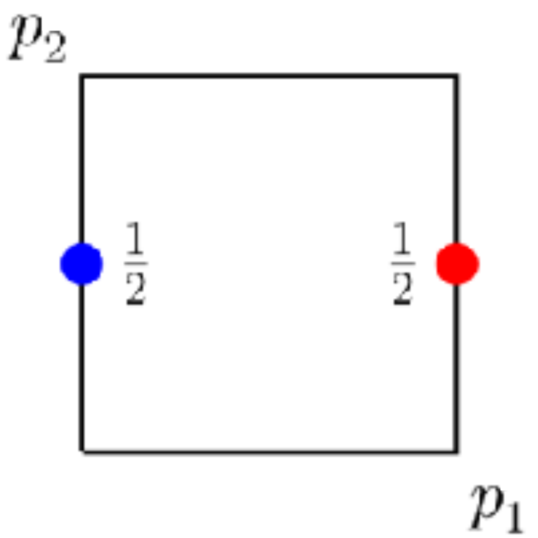
Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



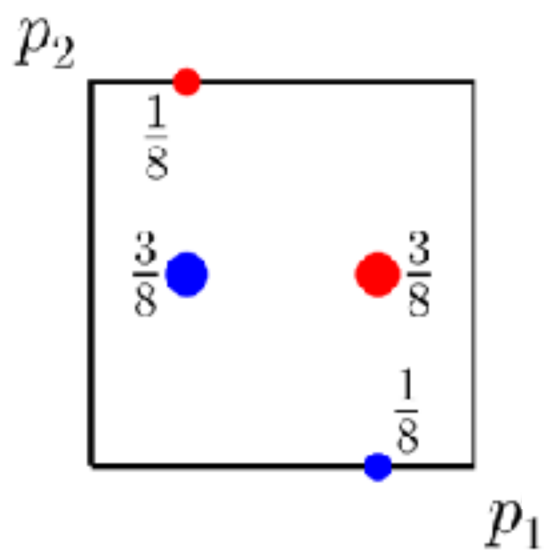
value $= -\dfrac{1}{32}$

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



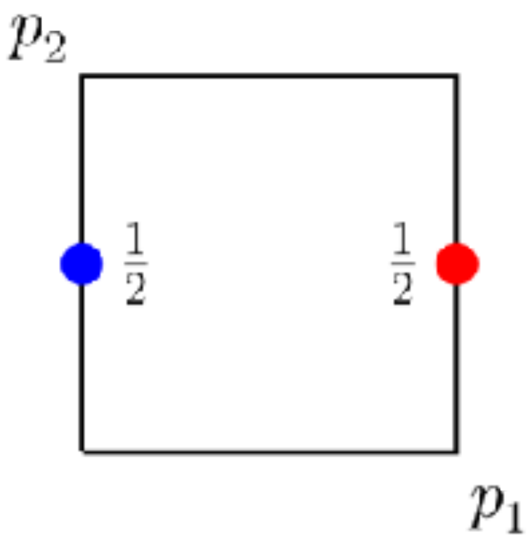
value $= -\dfrac{1}{32}$

HOW TO SOLVE?

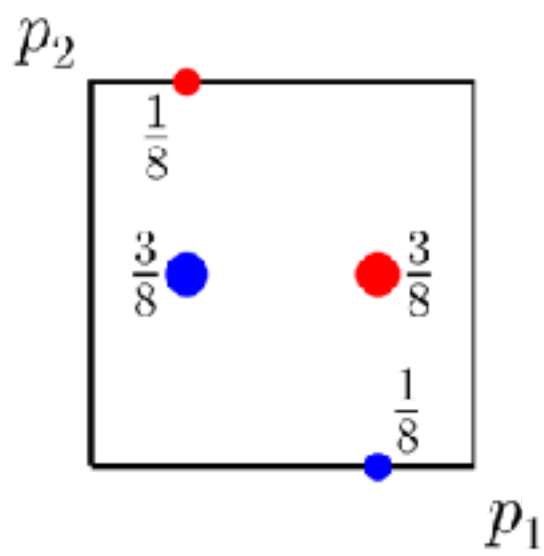
HILBERT-SPACE APPROACH:

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



value $= -\dfrac{1}{32}$

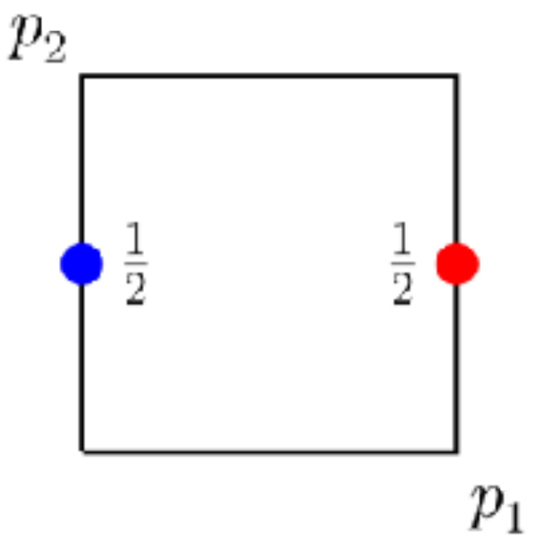
HOW TO SOLVE?

HILBERT-SPACE APPROACH:

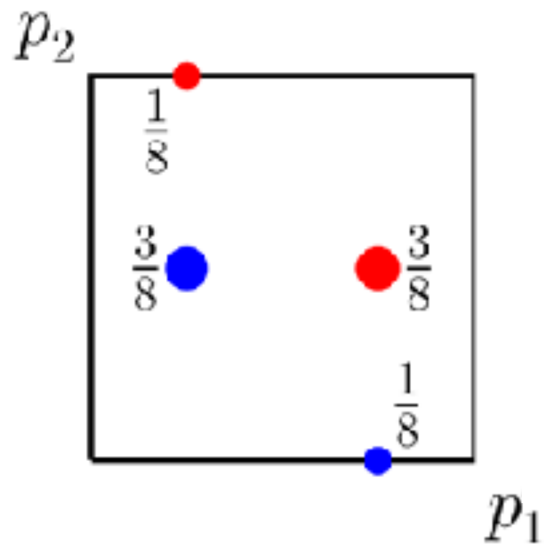
- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



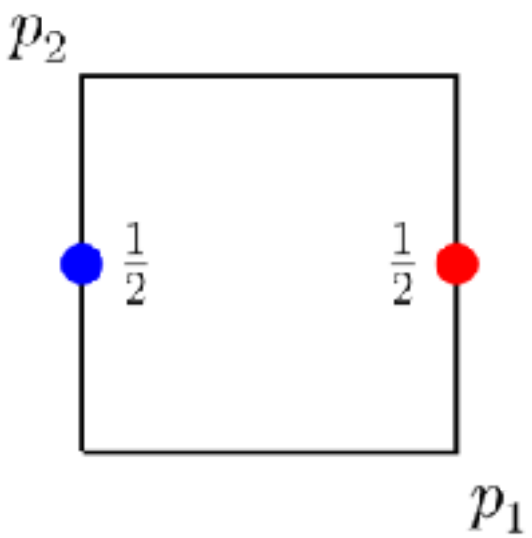
value $= -\dfrac{1}{32}$

HOW TO SOLVE? HILBERT-SPACE APPROACH:

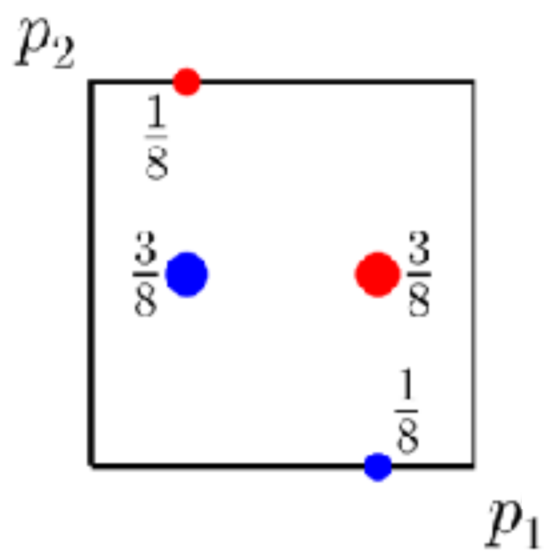
- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$
- $\xi \to \mathbb{E}[\xi \mid \mathscr{F}]$ is an orthogonal projection in L^2

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



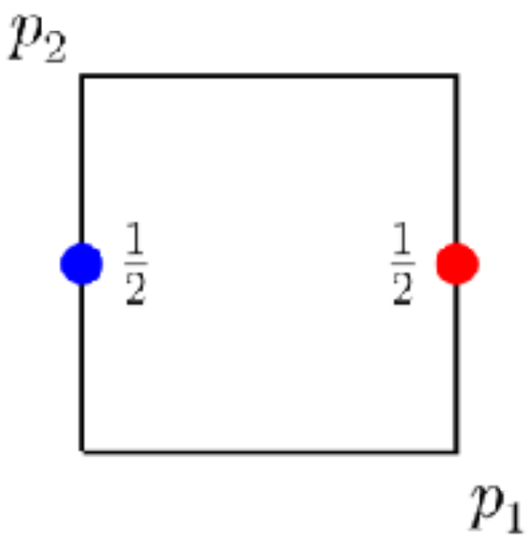
value $= -\dfrac{1}{32}$

HOW TO SOLVE? HILBERT-SPACE APPROACH:

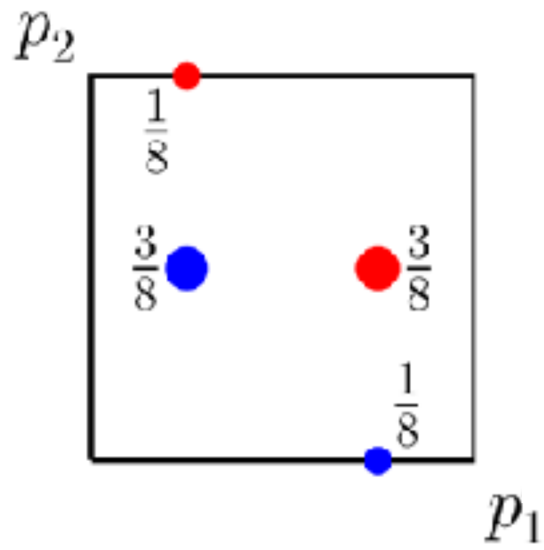
- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$
- $\xi \to \mathbb{E}[\xi \mid \mathscr{F}]$ is an orthogonal projection in L^2
- {all orthogonal projections of ξ}=sphere of radius $||\xi||/2$ centred at $\xi/2$

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



value $= -\dfrac{1}{32}$

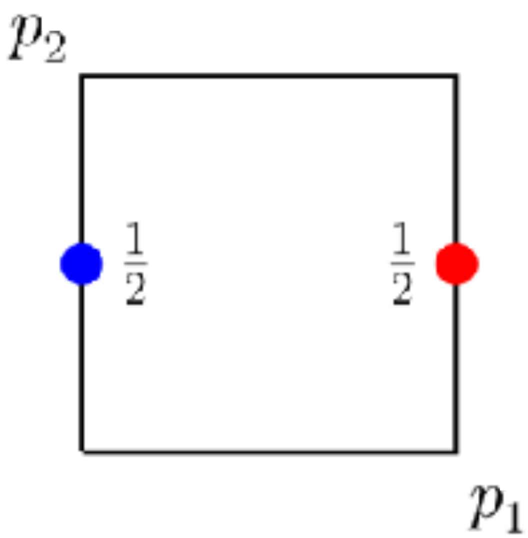
HOW TO SOLVE?

HILBERT-SPACE APPROACH:

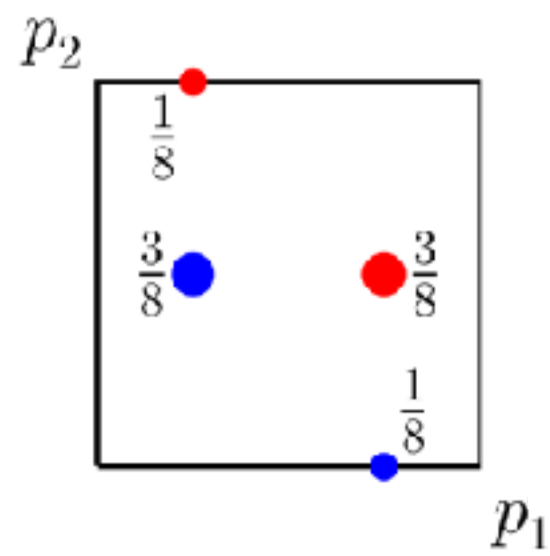
- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$
- $\xi \to \mathbb{E}[\xi \mid \mathscr{F}]$ is an orthogonal projection in L^2
- {all orthogonal projections of ξ}=sphere of radius $||\xi||/2$ centred at $\xi/2$
- express the quadratic objective through scalar products

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



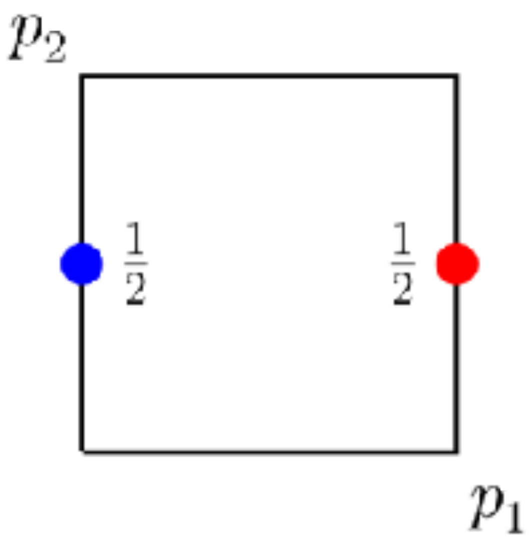
value $= -\dfrac{1}{32}$

HOW TO SOLVE? HILBERT-SPACE APPROACH:

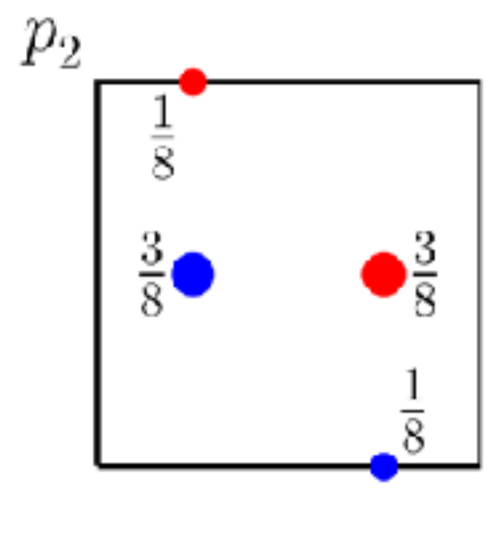
- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$
- $\xi \to \mathbb{E}[\xi \mid \mathscr{F}]$ is an orthogonal projection in L^2
- {all orthogonal projections of ξ}=sphere of radius $||\xi||/2$ centred at $\xi/2$
- express the quadratic objective through scalar products
- simple optimisation problem on the sphere

EXAMPLES: INDUCING CONFLICT FOR $p = 0.5$

Maximize $\mathbb{E}\,|p_1 - p_2|^2$



Minimize $\mathrm{cov}[p_1, p_2] = \mathbb{E}\big[(p_1 - 0.5)(p_2 - 0.5)\big]$



value $= -\dfrac{1}{32}$

HOW TO SOLVE? HILBERT-SPACE APPROACH:

- posterior = conditional expectation $p_i = \mathbb{E}[1_{\theta=1} \mid s_i]$
- $\xi \to \mathbb{E}[\xi \mid \mathscr{F}]$ is an orthogonal projection in L^2
- {all orthogonal projections of ξ}=sphere of radius $||\xi||/2$ centred at $\xi/2$
- express the quadratic objective through scalar products
- simple optimisation problem on the sphere

QUESTION Anything beyond quadratic objectives?

SUMMARY

- **N=1, N=2, N>2 differ significantly**
  - **Feasibility:** Quantitative Agreement Theorem (N=2), No-trade (N>2)
  - Optimal persuasion may require **infinite number of signals**
- **Open problems:**
  - indicator trades, continuum of signals, non-quadratic objectives

SUMMARY

- **N=1, N=2, N>2 differ significantly**
  - **Feasibility:** Quantitative Agreement Theorem (N=2), No-trade (N>2)
  - Optimal persuasion may require **infinite number of signals**
- **Open problems:**
  - indicator trades, continuum of signals, non-quadratic objectives

THANK YOU!

REFERENCES

- **Aumann, Maschler** Repeated games with incomplete information. *MIT Press*, 1995
- **Blackwell** Comparison of experiments. Second Berkeley *Symposium on Mathematical Statistics and Probability,* 1951
- **Dawid, DeGroot, Mortera** Coherent combination of experts' opinions. *Test*, 1995
- **Burdzy and Pitman** Bounds on the probability of radically different opinions. *Elect. Commun. in Prob.*, 2020
- **Ziegler** Adversarial bilateral information design. *Working Paper*, 2020
- **Levy, de Barreda, Razin** Persuasion with correlation neglect. *Working Paper*, 2020
- **Mathevet, Perego, Taneva** On information design in games. *JPE*, 2019
- **Morris** No trade and feasible joint posterior beliefs. *Working Paper*, 2020
- **Aumann** Agreeing to disagree. *The Annals of Statistics*, 1976
- **Kellerer** Duality theorems for marginal problems. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete,* 1984
- **Kamenica, Gentzkow** Bayesian persuasion. *AER*, 2011