Private Private Information

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EC'22

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- It is possible! We study tension between informativeness and privacy

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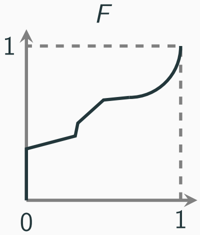

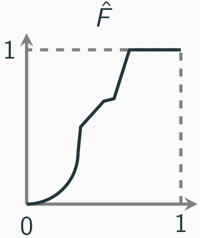

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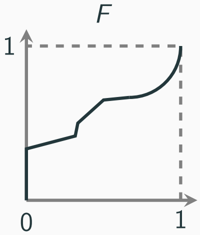

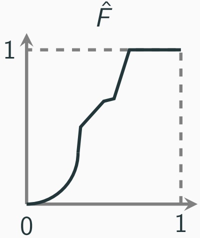

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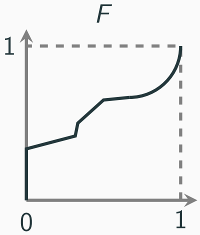

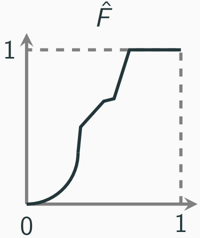

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Theorem 1

For $n = 2$, a private private info structure is Pareto optimal if and only if the belief distributions induced by s_1 and s_2 are conjugates.

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- for ≥ 3 states ω, there may be a continuum of optimal s_2

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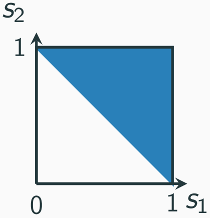

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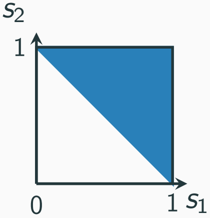

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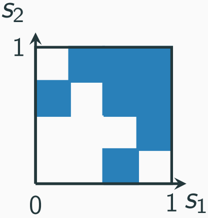

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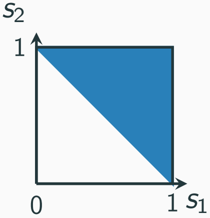

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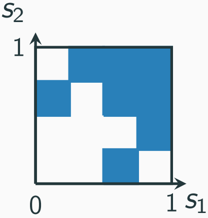

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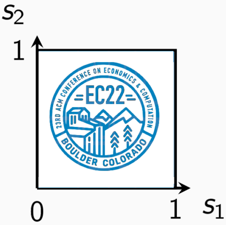

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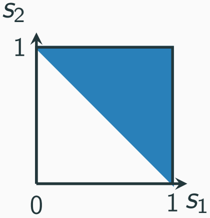

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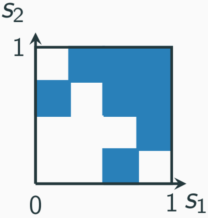

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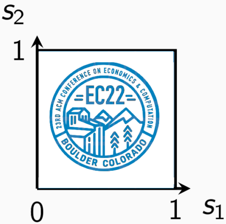

Proposition

Any private private info structure is equivalent to a structure associated with some $A \subseteq [0,1]^n$

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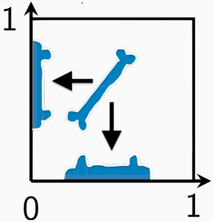
Tomography reconstructs objects from lower-dimensional projections



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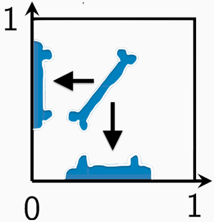

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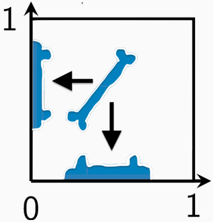

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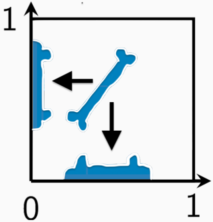

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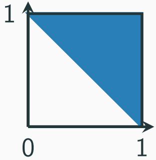

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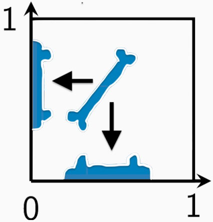

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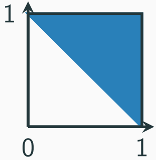

Corollary: characterization of Pareto Optimality via conjugates (Th 1)

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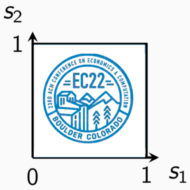

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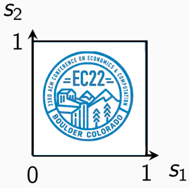

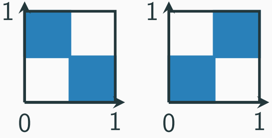

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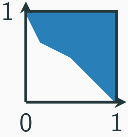

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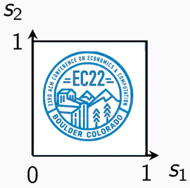
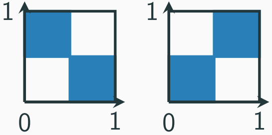
- Can **represent** all such info structures as subsets of $[0,1]^n$
- Pareto optimal private private info structures are associated with **sets of uniqueness**
  - For $n = 2$, a simple criterion of Pareto optimality: distributions of posteriors must be conjugate





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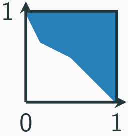
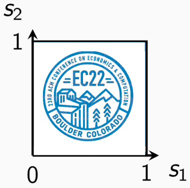


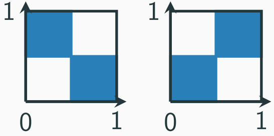
(Pareto optimal)

Summary

- **Private private information structures**: signals of different agents $(s_1, s_2, ..., s_n)$ are unconditionally independent
- Can **represent** all such info structures as subsets of $[0, 1]^n$

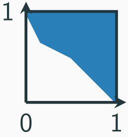


- Pareto optimal private private info structures are associated with **sets of uniqueness**
    - For $n = 2$, a simple criterion of Pareto optimality: distributions of posteriors must be conjugate



(not Pareto optimal)

Thank you!



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Other occurrences of private private signals

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  - Bergemann, Brooks, Morris *First-price auctions with general information structures:Implications for bidding and revenue* Econometrica 2017
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- Feasible joint distributions of posterior beliefs
  - Arieli, Babichenko, Sandomirskiy, Tamuz *Feasible joint posterior beliefs* Journal of Political Economy 2021