Private Private Information

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- A joint distribution $\mathbb{P}$ over $(\omega, s_1, ..., s_n)$ defines the **private information structure**

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  - s_1 contains info about s_2, so P2's info not fully private

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  - We focus on this tension and study how informative private private signals can be

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- Privacy $\simeq$ demographic parity w.r.t. a protected trait s_1 in fair machine learning
  - Barocas, Hardt, Narayanan. Fairness in machine learning. NeurIPS tutorial 2017

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- Feasible joint distributions of posterior beliefs
  - Arieli, Babichenko, Sandomirskiy, Tamuz *Feasible joint posterior beliefs* Journal of Political Economy 2021

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A private private structure is **Pareto optimal** if it is not dominated by another private private structure.

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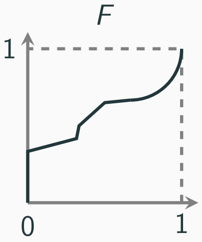

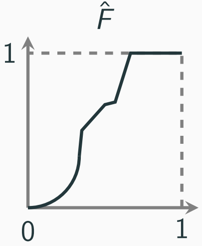

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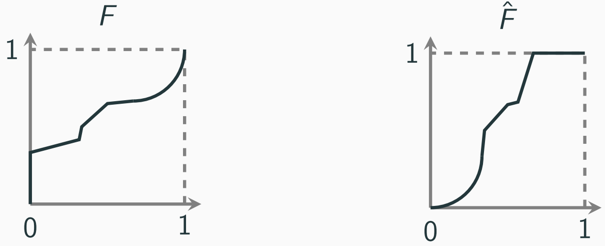

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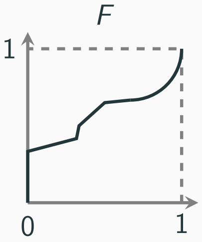

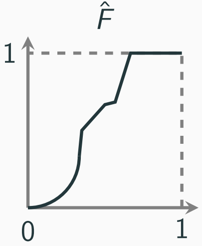

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Theorem 1

For $n = 2$, a private private info structure is Pareto optimal if and only if the belief distributions induced by s_1 and s_2 are conjugates.

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- For ≥ 3 states ω, there may be a continuum of optimal s_2

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- $\omega \in \{\ell, h\}$ is a job fit
- $s_1 \in \{y, n\}$ presence of a medical condition (yes/no)
- $\mathbb{P}(\omega = h) = \mathbb{P}(s_1 = y) = 1/2$
- $\mathbb{P}(\omega = h \mid s_1 = y) = \frac{3}{4}, \quad \mathbb{P}(\omega = h \mid s_1 = n) = \frac{1}{4}$
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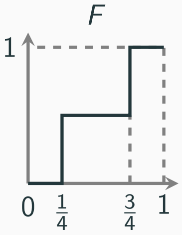
- 

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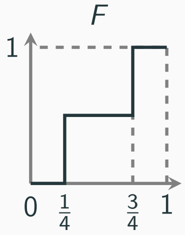
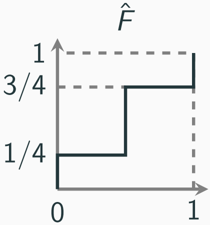
- 



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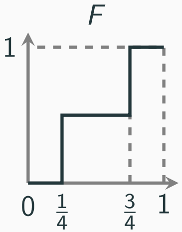
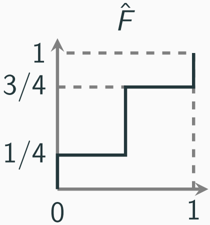
- $\mathbb{P}(\omega = h) = \mathbb{P}(s_1 = y) = 1/2$
- $\mathbb{P}(\omega = h \mid s_1 = y) = \frac{3}{4}, \quad \mathbb{P}(\omega = h \mid s_1 = n) = \frac{1}{4}$
- **Goal:** find s_2 that is informative of ω but independent of s_1

- 



- Optimal s_2 is trinary and induces posteriors (0, 1/2, 1) with probabilities (1/4, 1/2, 1/4)

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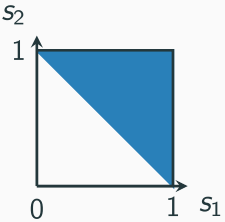

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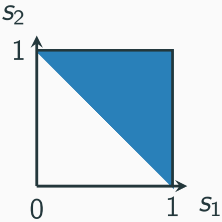

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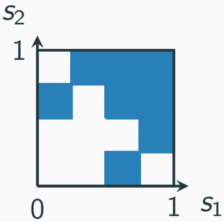

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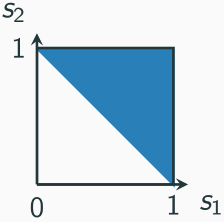

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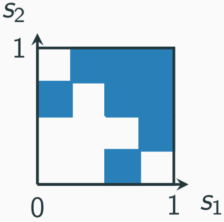

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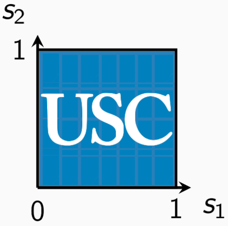

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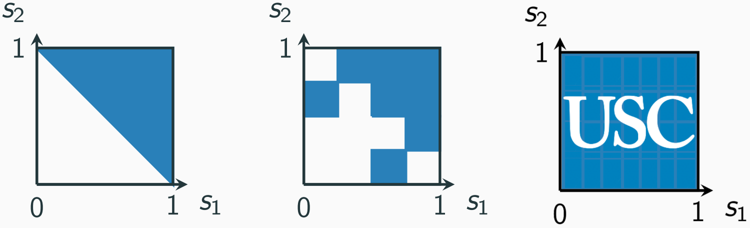

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Proposition

Any private private info structure is equivalent to a structure associated with some $A \subseteq [0,1]^n$

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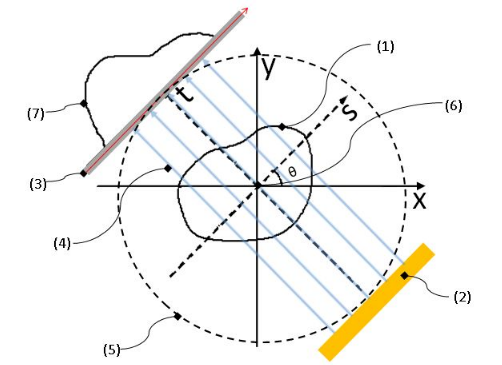

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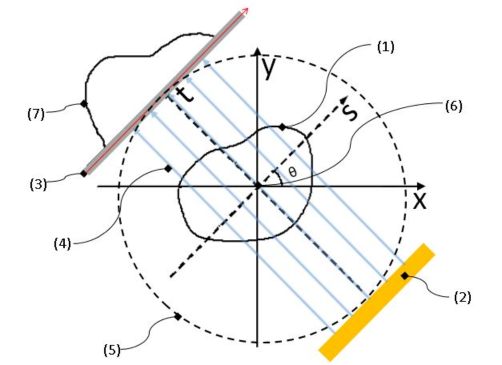
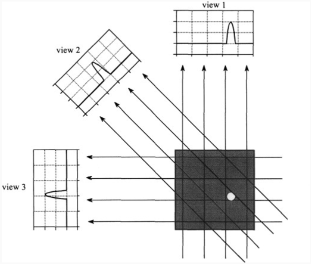


- Produces a lower-dimensional projection of the object by looking at how much x-ray is absorbed at different points

Tomography and Sets of Uniqueness

- Typically, must run x-ray from many different angles to get a good understanding of the object's geometry



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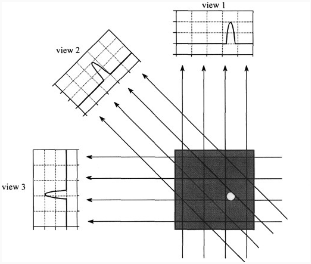

Definition

$A \subseteq [0,1]^n$ is a **set of uniqueness** if it is determined by its n coordinate projections, i.e., for any A' such that the uniform density on A and A' has the same one-dimensional marginals, $A' = A$ a.e. in $[0,1]^n$.

Pareto Optimality and Sets of Uniqueness

Theorem 2

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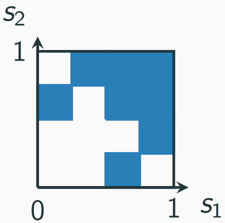

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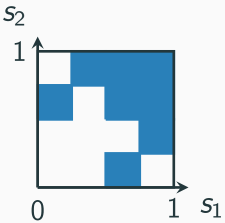
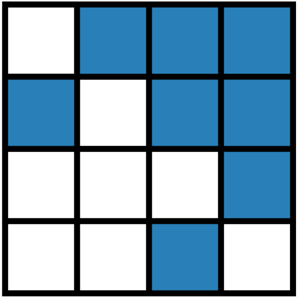


  is not Pareto optimal (s_2 can be replaced by a more informative trinary signal)
- Hence, the blue area not a set of uniqueness. Let's check!

A Puzzle!

Problem for a newspaper puzzle column: is there another coloring of the 4x4 grid that preserves all column-wise and row-wise counts of colored squares?



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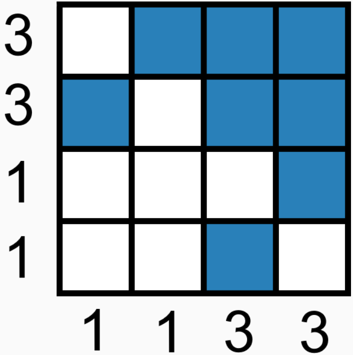

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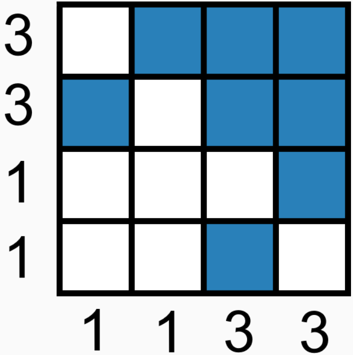

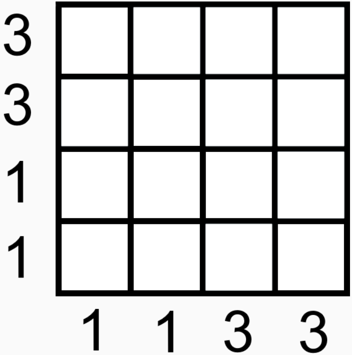

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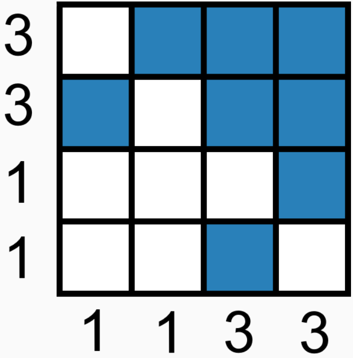

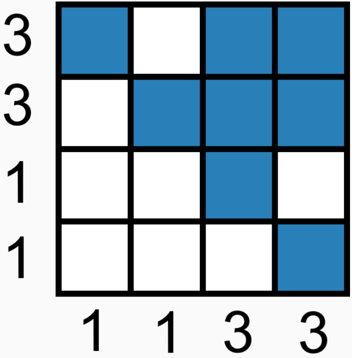

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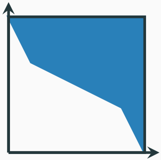

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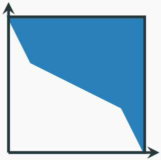

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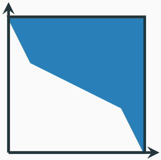

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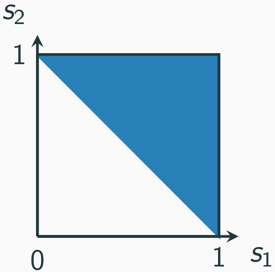

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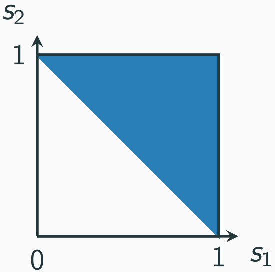

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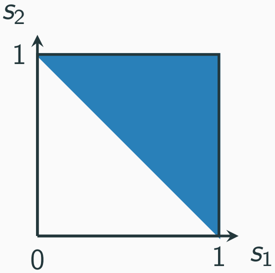

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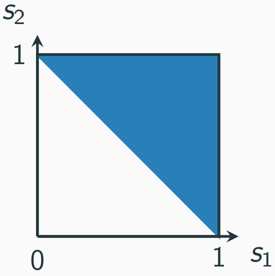

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Connecting Pareto Optimality with Tomography

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For any $n \geq 2$, a private private info structure is Pareto optimal $\Leftrightarrow$ it is equivalent to a structure associated with a set of uniqueness $A \subseteq [0, 1]^n$.

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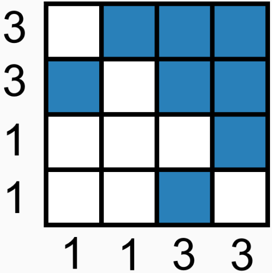
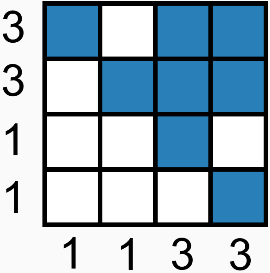
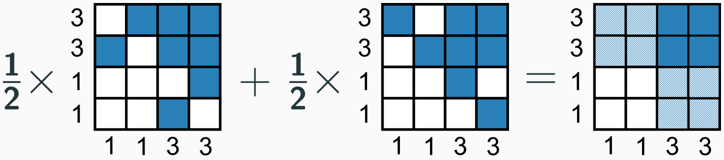




Not a Set of Uniqueness ⇒ Strictly Dominated



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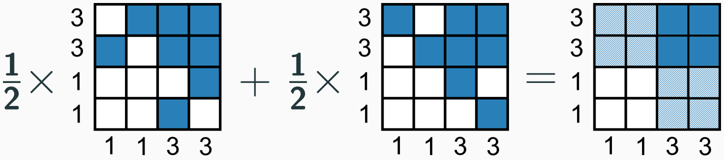

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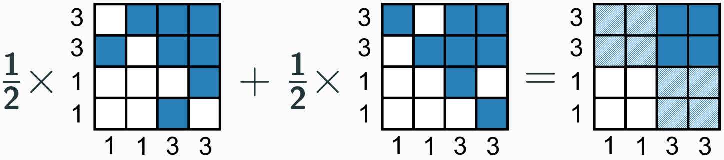

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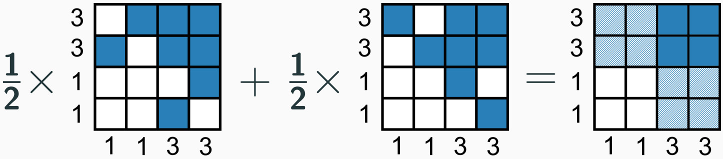

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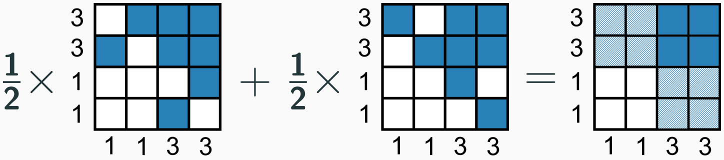

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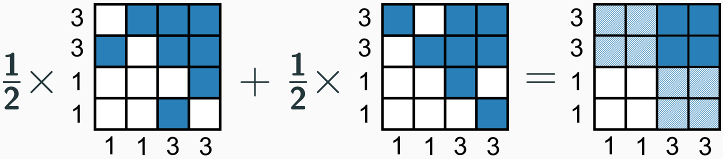

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- Reveal the coin toss to the first player □

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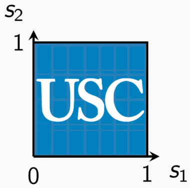
- Can **represent** all such info structures as subsets of $[0, 1]^n$



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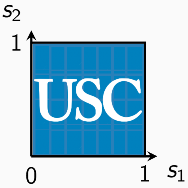
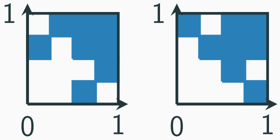
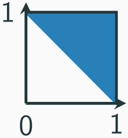
- Can **represent** all such info structures as subsets of $[0, 1]^n$
- Pareto optimal private private info structures are associated with **sets of uniqueness**





(not Pareto optimal)



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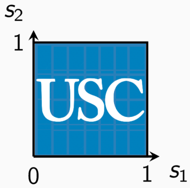

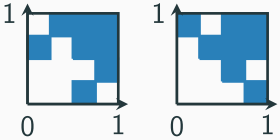

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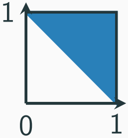

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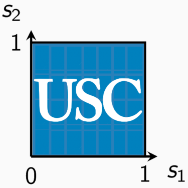

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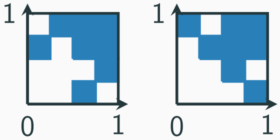

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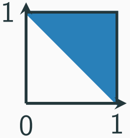

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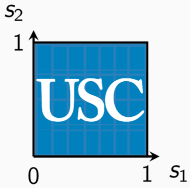

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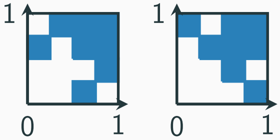

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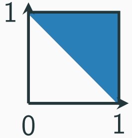

Thank you!