Beckmann's approach to multi-item multi-bidder auctions

arXiv:2203.06837

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For $n \geq 1$ bidders with additive utilities over $m \geq 1$ items

$$\max_{\text{BIC IR mechanisms}} \text{Revenue} = \min_{\text{transport flows}} \text{Cost}$$

- formal statement later
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For given geographical distribution of production $\pi_+ \in \Delta(X)$ and consumption $\pi_- \in \Delta(X)$, $X \subset \mathbb{R}^d$, find least costly way of transportation

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Related literature

- **Econ applications of optimal transport**
    - Monge-Kantorovich: Daskalakis et al. (2017), Kleiner, Manelli (2019), Boerma et al.(2021), Chiapporiet et al. (2010), Galichon (2021), Steinerberger, Tsyvinski (2019), Gensbittel (2015), Arieli et al.,(2022), Guo, Shmaya (2021)
    - Beckmann: Fajgelbaum, Schaal (2020), Allenand, Arkolakis (2014), Santambrogio (2015)
- **Non-transport duality in auction design** Giannakopoulos, Koutsoupias (2018), Cai et al. (2019), Bergemann et al. (2016)
- **Simple mechanisms with good revenue guaratees** Hart, Reny (2019), Haghpanah, Hartline (2021), Babaioff et al. (2020,2021), Hart, Nisan (2017), Jehiel et al. (2007), Yao (2017), and many more...
- **Majorization in economics** Hart and Reny (2015), Kleiner et al. (2021), Arieli et al. (2019), Candogan, Strack (2021), Nikzad (2022)

Known results: $m \geq 2$ goods, $n = 1$ agent

- agent with values $v = (v_1, \ldots, v_m) \sim \rho(v)\,\mathrm{d}v$ and additive utilities
- **Goal:** maximize revenue over BIC IR mechanisms
- **Rochet-Chone approach:** mechanisms $\Leftrightarrow$ interim utility functions

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$$\text{optimal revenue} = \max_{\substack{\text{convex monotone } u \\ u(0)=0, \\ 1-\text{Lipshitz}}} \int_{\mathbb{R}_+^m} \Big(\langle \partial u(v), v \rangle - u(v) \Big) \rho(v)\,\mathrm{d}v$$

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$$\big[\text{integrating by parts}\big]$$

$$= \max_{\substack{\text{convex } u \\ u(0)=0, \\ 1-\text{Lipshitz}}} \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi,$$

where $\mathrm{d}\psi = \big((m+1)\rho(v) + \sum_{j=1}^m v_i \partial_{v_i}\rho\big)\,\mathrm{d}v$ (signed measure!)

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$$\text{optimal revenue} = n \cdot \max_{\substack{\text{convex monotone } u \\ u(0)=0, \\ \partial_{v_i} u(v) \preceq z^{n-1}\,\forall i \\ z \sim \mathrm{Uniform}([0,1])}} \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi$$

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  - **reduction:** n-agent mechanism $\to$ 1-agent reduced form
  - **characterization of feasible reduced forms via majorization:** $m = 1$ proved by Hart and Reny[1], equivalent to Border's theorem

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What is the dual?

Beckmann: $\mathrm{B}_\rho(\pi, \Phi) = \min_{f\,:\,\mathrm{div}[\rho \cdot f] + \pi = 0} \int_{\mathbb{R}_+^m} \Phi(f(v)) \cdot \rho(v)\,\mathrm{d}v$

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Theorem (strong duality)

$$\text{optimal revenue} = n \cdot \min_{\substack{\pi \succeq \psi \\ \varphi_i \text{ on } \mathbb{R}_+ \text{ s.t.} \\ \text{convex, monotone, } \varphi_i(0)=0}} \left[\mathrm{B}_\rho\Big(\pi, \Phi\Big) + \sum_{i=1}^m \int_0^1 \varphi_i\left(z^{n-1}\right)\,\mathrm{d}z \right],$$

where $\Phi(f) = \sum_{i=1}^m \varphi_i^*(|f_i|)$ and $\varphi_i^*(y) = \sup_x \langle x, y \rangle - \varphi_i(x)$

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Theorem (Santambrogio (2015))

$$\mathrm{B}_\rho\Big(\pi,\ \|\cdot\|_1\Big) = \min_{\substack{\text{positive measures } \gamma \\ \text{with marginals } \pi_+, \pi_-}} \int \|v - v'\|_1 \, \mathrm{d}\gamma(v, v')$$

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Corollary: duality by Daskalakis et al. (2017)

Applications

Strong duality $\Rightarrow$ complementary slackness conditions

- allow to **disprove** optimality
  - **Example:** For $\rho(v) = \rho_1(v_1) \cdot \ldots \cdot \rho_m(v_m)$, selling separately is never optimal[1]
- help to **guess** an explicit solution and to **prove optimality**
  - **Example:** For $n = 1$ and $m = 2$ i.i.d. uniform items, selling each for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$ is optimal[2]

[1] P. Jehiel, M.Meyer-Ter-Vehn, B.Moldovanu (2007) Mixed bundling auctions JET
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Strong duality $\Rightarrow$ complementary slackness conditions

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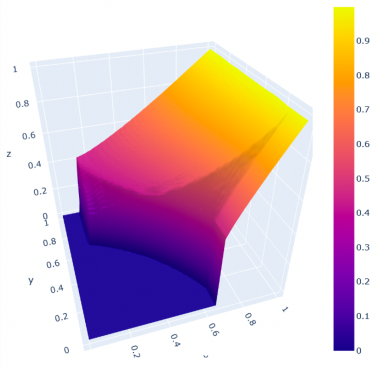
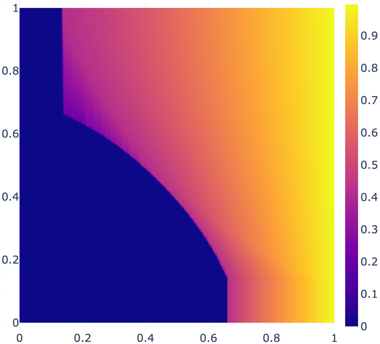
Pictures for dessert: 2 bidders, 2 i.i.d. uniform items

Probability to receive the first item as a function of bidder's values (v_1, v_2) in the optimal auction (about algorithm):





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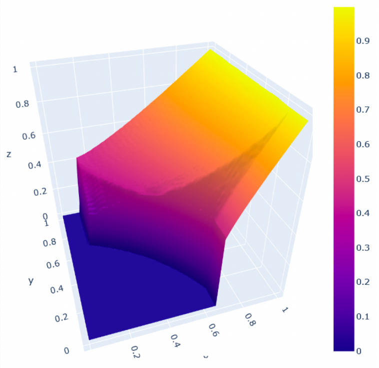

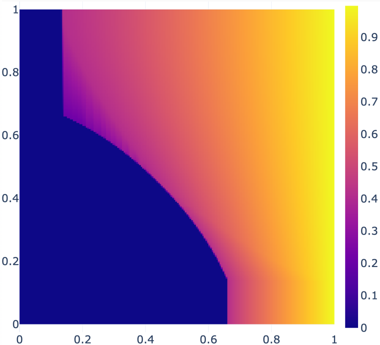

Thank you!

Complementary slackness conditions

back to theorem

Optimal u^{opt}, functions φ_i^{opt}, measure π^{opt}, and vector field f^{opt} satisfy:

$$\int u^{\mathrm{opt}}(v)\,\mathrm{d}\psi(v) = \int u^{\mathrm{opt}}(v)\,\mathrm{d}\pi^{\mathrm{opt}}(v)$$

$$f_i^{\mathrm{opt}}(v) \in \partial\varphi_i^{\mathrm{opt}}\left(\frac{\partial u^{\mathrm{opt}}}{\partial v_i}(v)\right)$$

$$\int \varphi_i^{\mathrm{opt}}\left(\frac{\partial u^{\mathrm{opt}}}{\partial v_i}(v)\right)\rho(v)\,\mathrm{d}v = \int_0^1 \varphi_i^{\mathrm{opt}}\left(z^{n-1}\right)\,\mathrm{d}z$$

Algorithmic ideas

back to simulations

- **Automated mechanism design:** revenue maximization is an LP, let's feed it to an LP solver; Sandholm (2003)
- **Curse of dimensionality:** If each of n agents can have q different values for each of m items $\Rightarrow$ the dimension $\sim \left(q^n\right)^m$
  - intractable for ($m = 2,\ q = 100\ n = 2$) or for ($m = 2\ q = 10\ n = 4$)
  - deep neural networks improve the bounds; Dutting et al. (2019)
- **How to avoid:**

$$R_{n,m}(\rho) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_i} u(v) \preceq z^{n-1}}} n \cdot \int_{\mathbb{R}_+^m} u(v)\,\mathrm{d}\psi(v)$$

  - Pros: dependence on n is killed; Cai et al.(2012), Alaei et al. (2019)
  - Cons: non-linear program
- **Linearization via transport:**
  - μ on $[0,1]$ majorizes ν if and only if there is γ on $[0,1]^2$ with marginals μ on y and ν on x and such that $\int y\,\mathrm{d}\gamma(y \mid x) \geq x$ for γ-almost all x
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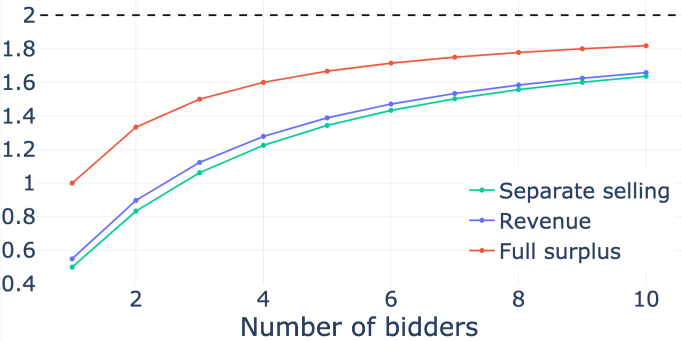
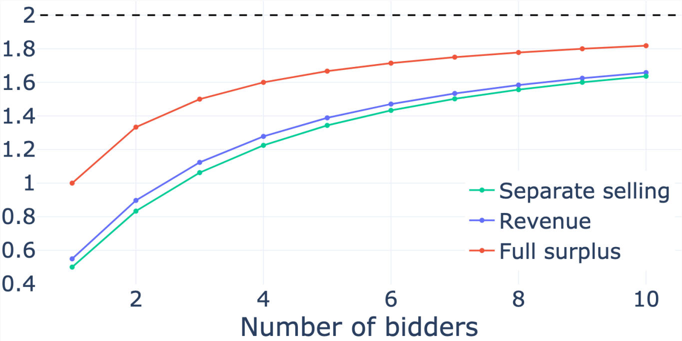


Revenue as a function of the number of bidders n for two items with i.i.d. values uniform on $[0,1]$. Graphs from bottom to top: selling separately (light-green), selling optimally (blue), full surplus extraction (red), limit for $n \to \infty$ (the dashed line).

Remark: For $n = 2$, selling optimally improves upon selling separately by 5%

Revenue

back to algorithmic ideas



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