Persuasion as Transportation*

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Abstract

We consider a model of Bayesian persuasion with one informed sender and several uninformed receivers. The sender can affect receivers' beliefs via private signals and the sender's objective depends on the combination of induced beliefs.

We reduce the persuasion problem to the Monge-Kantorovich problem of optimal transportation. Using insights from optimal transportation theory, we identify several classes of multi-receiver problems that admit explicit solutions, get general structural results, derive a dual representation for the value, and generalize the celebrated concavification formula for the value to multi-receiver problems.

1 Introduction

Actions taken by economic agents depend on information they have access to and thus more informed agents can use their information advantage to affect the actions of less informed ones by disclosing the available information selectively. Such a strategic information disclosure is called persuasion. In the archetypal problem of Bayesian persuasion by Kamenica and Gentzkow (2011), an informed sender aims to affect uninformed receiver's beliefs by sending a noisy signal. This model has become a standard for understanding information-related phenomena in various economic problems such as advertising, market signalling, legal disputes, financial disclosure, and many others. Presence of explicit solutions which can be constructed via the concavification technique of Aumann and Maschler (1995) contributes to the popularity of this model.

The assumption that the sender interacts with a single receiver can often be restrictive. For example, electronic marketplaces are governed by recommendation systems that are able to send

*We are grateful to Omer Tamuz for discussions that have inspired this work and to Emir Kamenica for useful comments.

†Technion, Haifa (Israel). Itai Arieli has been supported by the Israel Science Foundation (grant #2030524).

‡Technion, Haifa (Israel). Yakov Babichenko has been supported by BSF grants #2018397 and #2021680.

§Caltech, Pasadena (USA). Fedor Sandomirskiy has been supported by the Linde Institute at Caltech and the National Science Foundation (grant CNS 1518941).

"recommendations" individually tailored to each recipient, i.e., private signals. However, private multi-receiver Bayesian persuasion is challenging. The source of difficulty is that, in addition to deciding what information to disclose to each receiver (which boils down to specifying individual belief distributions, as in the single-receiver case), the sender must also decide on how to couple (i.e., to correlate) the information across receivers. As pointed out by Dawid et al. (1995); Arieli et al. (2021a), the set of feasible couplings of individual belief distributions has a complex structure making optimal multi-receiver persuasion notoriously difficult.

The classic Monge-Kantorovich problem of optimal transportation consists in finding the least costly way to transport a produced commodity for given spacial distributions of supply and demand. More generally, it can be thought of as a problem of finding a coupling between given distributions optimizing a certain objective function. This interpretation explains why optimal transportation problems emerge in diverse economic, mathematical, and statistical contexts seemingly having no connection to transportation.

Coupling of beliefs is the bottleneck in understanding private multi-receiver persuasion. This observation motivates finding a formal connection between persuasion and transportation to handle the belief-coupling problem via transportation tools.

Our contribution

We demonstrate that private multi-receiver persuasion can be reduced to a problem of optimal transportation, moreover, methods from optimal transportation enabled by this connection prove to be useful to study persuasion problems.

The reduction is established in Theorem 1. The essence of the result is that persuasion can be represented as nested optimization: first, over individual belief distributions and then over possible couplings for fixed individual distributions. The internal coupling problem turns out to be the standard optimal transportation one. The external problem takes a simple form as feasibility constraints on individual belief distributions are easy to describe. The main insight is that, counter-intuitively, conditional distributions of beliefs given the realized state are substantially easier to handle than unconditional ones. This insight allows us to bypass the non-tractability demonstrated by Arieli et al. (2021a).

The internal transportation problem absorbs the difficulty of the original persuasion problem. For some classes of sender's objectives, one can make use of optimal transportation tools to solve the internal problem explicitly and, as a result, the persuasion problem itself. We illustrate this approach for one-state persuasion, where the sender's utility is zero for all states except one (Section 3.1) and for supermodular utilities (Section 3.2.1).

Duality plays the central role in optimal transportation theory. Using the connection between persuasion and transportation, we obtain an analog of the Kantorovich–Rubinstein duality for persuasion (Theorem 2). The dual representation for the sender's optimal value extends the familiar concavification formula for the single-receiver problem to the multi-receiver case (Theorem 3). Specifically, we show that the value can be expressed as the minimum over all functions that are pointwise above the sender's utility and for which revealing no information is optimal for all priors.[1] Finally, we demonstrate how the dual problem can be used to construct explicit solutions to the primal one.

1.1 Related literature

Optimal ways to persuade multiple receivers via private signals are known only for particular sender's objectives and/or strong restrictions on receivers' action sets. The main obstacle is the complex structure of the set of feasible joint belief distributions (those distributions that the sender can induce) as indicated by Dawid et al. (1995); Arieli et al. (2021a); Mathevet et al. (2020); Arieli et al. (2021b); He et al. (2021). Related feasibility questions were studied by Herings et al. (2020); Ziegler (2020); Morris (2020); Brooks et al. (2019); Babichenko et al. (2021). Mathematical literature Burdzy and Pitman (2020); Burdzy and Pal (2021); Cichomski (2020); Cichomski and Osekowski (2021) provides some tight bounds on feasible distributions, where they called coherent distributions. These tight bounds can be converted into solutions to particular persuasion problems. Persuasion simplifies dramatically if receivers have only a few actions. Arieli and Babichenko (2019) solve the problem for binary actions and sub/supermodular objectives. In general, for a few actions, one can identify signals with action recommendations satisfying incentive-compatibility constraints (also known as obedience or straightforwardness Kamenica and Gentzkow (2011)) and obtain the optimal policy as a solution to a linear program capturing Bayesian correlated equilibria as in Bergemann and Morris (2016, 2019); Taneva (2019). Our results are not sensitive to the cardinality of action sets and are applicable in the case of a continuum of actions.

A connection to optimal transportation is known in a variety of economic settings, e.g., monopoly pricing Daskalakis et al. (2017), labor market sorting Boerma et al. (2021), econometrics Galichon (2021), and many others surveyed by Ekeland (2010); Carlier (2012); Galichon (2016). This connection is fruitful as it always brings new tools — such as the Kantorovich-Rubinstein duality — from mathematical theory of transportation to the problem of interest. The modern mathematical theory is surveyed by Bogachev and Kolesnikov (2012); Guillen and McCann (2013) and comprehensively presented in books Santambrogio (2015); Villani (2009).

In Bayesian persuasion and, more generally, information design, a connection to optimal trans-

[1]For a single receiver, the set of utility functions such that revealing no information is optimal for all priors is precisely the set of concave functions.

port has not been known and was mentioned as an open problem by Dworczak and Martini (2019). However, in the single-receiver case, several papers establish duality remotely similar to the transportation one. Kolotilin (2018) relied on infinite-dimensional linear programming and found the dual to persuading partially-informed receiver taking a binary action. Dworczak and Martini (2019) proposed a dual approach based on price functions to problems where the sender's utility depends on the posterior mean; see also Dizdar and Kováč (2020). Dworczak and Kolotilin (2019) adapt the duality approach of Dworczak and Martini (2019) to handle the general sender's objectives. The duality that we find in the multi-receiver setting can be seen as an extension of their single-receiver result; see Section 4 for a detailed comparison.

2 Model

2.1 Persuasion

A *Bayesian persuasion problem* is given by a collection

$$B = \Big(\Omega,\ p \in \Delta(\Omega),\ N,\ v\,:\,\Omega \times \big(\Delta(\Omega)\big)^N \to \mathbb{R}\Big).$$

Here Ω is a finite set of states and a random state $\omega \in \Omega$ is drawn according to a distribution $p = (p(\omega))_{\omega\in\Omega} \in \Delta(\Omega)$ with full support. We refer to p as the *prior distribution.*

The sender observes the realized state ω and can selectively reveal some information about ω to a group of n receivers $N = \{1, 2, \ldots, n\}$, who do not observe the realization of ω but are aware of the prior distribution. The information is revealed via an information structure with private signals, which is defined below. The goal of the sender is to maximize her expected utility $v^\omega(x_1, x_2, \ldots, x_n)$, which depends on the state ω and on the combination of posterior beliefs[2] $x_1, x_2, \ldots x_n$ of all the receivers about the state; the function v is assumed to be measurable.

An *information structure* $I = \big((S_i)_{i\in N}, \pi(\cdot \mid \omega)\big)$ is composed of sets of signals S_i for each receiver $i \in N$ and a joint distribution of signals $\pi(\cdot \mid \omega) \in \Delta(S_1 \times \cdots \times S_n)$ conditional on each possible realization of the state ω. The sets of signals can be arbitrary measurable spaces, i.e., sets equipped with sigma-fields. It is assumed that the sender selects an information structure before observing the state and commits to drawing signals $(s_1, \ldots, s_n)$ according to the distribution $\pi(\cdot \mid \omega)$ once she observes ω.

[2]Utility functions depending on the profile of receivers' beliefs about the state arise as indirect utilities if each receiver i has action set A_i and the receiver's action is a function of her belief about the state, $a_i = a_i(x_i)$. This is the case in the first-order persuasion model of Arieli et al. (2020) where each receiver's utility depends on receiver's own action and the state only. More generally, higher-order beliefs do not affect receivers' choices if receivers play a simple game in the sense of Börgers and Li (2019).

Combined with the prior $p \in \Delta(\Omega)$, an information structure I induces the joint distribution $\mathbb{P} = \mathbb{P}_I$ of the state and signals $(\omega, s_1, \ldots, s_n)$. Each receiver i is aware of the prior p and the information structure I chosen by the sender. Hence, having received her signal s_i, the receiver i can compute her posterior belief $x_i \in \Delta(\Omega)$ about the state, i.e., $x_i(w) = \mathbb{P}_I(\omega = w \mid s_i)$, $w \in \Omega$. The posterior belief is defined for almost all realizations of signals. For finite sets of signals, it can be computed by the familiar Bayes formula:

$$x_i(w) = p(w) \cdot \frac{\pi(s_i \mid w)}{\sum_{w' \in \Omega} p(w') \cdot \pi(s_i \mid w')}. \tag{1}$$

The persuasion problem is to maximize the expected utility $\mathbb{E}_I[v^\omega(x_1, \ldots, x_n)]$ over all information structures I. The optimal value of the the objective is called the value of the persuasion problem B:

$$\mathrm{Val}[B] = \sup_I \mathbb{E}_I\Big[v^\omega(x_1, x_2 \ldots, x_n)\Big]. \tag{2}$$

Note that at this point we assume neither boundedness nor continuity of v and, hence, the value may equal $+\infty$ or the optimal information structure may fail to exist (this is why the value is defined using sup instead of max). As we will see later, the existence of the optimal information structure is guaranteed under the standard assumption of upper semicontinuity.

2.2 Transportation

Suppose we are given a finite set X_1 of locations, where the same homogeneous good is produced, and a finite set X_2 of consumers. A distribution λ_1 over X_1 represents the amount produced at each location and a distribution λ_2 over X_2 specifies the demand of each consumer. The cost of transporting a unit amount of the good from a location x_1 to a consumer x_2 is $c(x_1, x_2)$, where $c\colon X_1 \times X_2 \to \mathbb{R}$ is a given cost function. A transportation plan μ is given by an $X_1 \times X_2$ matrix, where $\mu(x_1, x_2) \geqslant 0$ is the amount transported from x_1 to x_2; a plan is feasible if supply meets demand, i.e., $\sum_{x_2 \in X_2} \mu(x_1, x_2) = \lambda_1(x_1)$ and $\sum_{x_1 \in X_1} \mu(x_1, x_2) = \lambda_2(x_2)$ for all x_1 and x_2. The classic Monge-Kantorovich transportation problem is to find a feasible plan μ with the minimal total transportation cost.

More generally, instead of two sets X_1 and X_2 there is an arbitrary number of them X_i, $i \in N = \{1, 2, \ldots, n\}$ and X_i are arbitrary sets, not necessary finite, each equipped with a sigma algebra. For presentation purposes, it is convenient to consider a maximization objective instead of a minimization one. Thus a problem of optimal transportation is given by a measurable utility function v on $X_1 \times \ldots \times X_n$ and a collection of probability measures $\lambda_i \in \Delta(X_i)$ for each $i \in N$. Let $\mathcal{M}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the set of feasible transportation plans; it consists of probability measures μ on $X_1 \times \ldots \times X_n$ such that the marginal of μ on X_i equals λ_i for each $i \in N$. The goal is to

$$\text{maximize} \quad \int_{X_1 \times \ldots \times X_n} v(x)\, \mathrm{d}\mu(x) \quad \text{over} \quad \mu \in \mathcal{M}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We denote the value of the transportation problem by

$$T_v\big[(\lambda_i)_{i\in N}\big] = \sup_{\mu\in\mathcal{M}((\lambda_i)_{i\in N})} \int_{X_1\times\ldots\times X_n} v(x)\,\mathrm{d}\mu(x).$$

3 Persuasion as transportation

In this section, we show that the persuasion problem can be reduced to a transportation problem.

Consider a persuasion problem $B = \Big(\Omega, p, N, v\Big)$ and define a family of transportation problems indexed by $\omega \in \Omega$. In these transportation problems, the sets $X_1, X_2, \ldots, X_n$ coincide with $\Delta(\Omega)$ and the utility is v^ω. The marginals have to satisfy the requirement of admissibility that we are about to define.

Denote by $\Delta_p(\Delta(\Omega))$ the set of distributions on $\Delta(\Omega)$ with mean p, i.e., $\lambda \in \Delta_p(\Delta(\Omega))$ if $\int_{\Delta(\Omega)} x(\omega)\mathrm{d}\lambda(x) = p(\omega)$ for all $\omega \in \Omega$. By the well-known splitting lemma Aumann and Maschler (1995); Blackwell (1951), the set of all belief distributions of one agent that can be induced by some information structure is precisely $\Delta_p(\Delta(\Omega))$. Moreover, a distribution of beliefs $\lambda \in \Delta_p(\Delta(\Omega))$ uniquely determines the distribution of beliefs conditional on state ω. This conditional distribution denoted by λ^ω can be found using the following equality of the Radon-Nikodym derivatives:

$$\frac{\mathrm{d}\lambda^\omega}{\mathrm{d}\lambda}(x) = \frac{x(\omega)}{p(\omega)}, \qquad \text{for all } x \in \Delta(\Omega) \text{ and } \omega \in \Omega. \tag{3}$$

Definition 1. *An $|\Omega|$-tuple of distributions $(\lambda^\omega)_{\omega\in\Omega}$ is called* admissible *if there exists λ that induces λ^ω conditional on ω for every $\omega \in \Omega$, i.e., the identity (3) holds. A collection $(\lambda_i^\omega)_{i\in N,\omega\in\Omega}$ is called* admissible marginals *if the tuple $(\lambda_i^\omega)_{\omega\in\Omega}$ is an admissible $|\Omega|$-tuple for every $i \in N$.*

Theorem 1. *For a persuasion problem B, the value can be represented as follows:*

$$\mathrm{Val}[B] = \sup_{\substack{\text{admissible marginals} \\ (\lambda_i^\omega)_{i\in N,\omega\in\Omega} \subset \Delta(\Delta(\Omega))}} \sum_{\omega\in\Omega} p(\omega)\cdot T_{v^\omega}\big[(\lambda_i^\omega)_{i\in N}\big]. \tag{4}$$

Moreover, if the utility function v is upper semicontinuous, the optimal marginals, as well as the optimal transportation plans, exist and sup *can be replaced by* max.

In other words, to compute the value of a persuasion problem, we can fix some admissible marginals, solve a family of transportation problems indexed by the state, average the obtained values over the prior, and then optimize the result over admissible marginals. Theorem 1 is proved in Appendix A.1 where we also demonstrate how to construct an optimal information structure based on optimal marginals and transportation plan. Here we explain the structural properties of multi-receiver persuasion enabling this nested representation.

Ideas behind Theorem 1. Instead of the maximization over information structures in (2), one can maximize over all joint distributions of the state ω and posterior beliefs $x_1, \ldots, x_n$ that can be induced by some information structure I. Since the distribution of ω equals the prior p, we only need to know conditional distributions of beliefs given the state to reconstruct the whole distribution. We say that $(\mu^\omega)_{\omega\in\Omega} \subset \Delta\big(\Delta(\Omega) \times \ldots \times \Delta(\Omega)\big)$ are *feasible conditional distributions of beliefs* if there exists an information structure I such that the joint distribution of beliefs $x_1, \ldots, x_n$ given the state ω equals μ^ω for each of the states; the set of all such collections $(\mu^\omega)_{\omega\in\Omega}$ depends on the prior p and is denoted by $\mathcal{F}_p$.

We conclude that the value of the persuasion problem admits the following representation

$$\mathrm{Val}[B] = \sup_{(\mu^\omega)_{\omega\in\Omega}\in\mathcal{F}_p} \sum_{\omega\in\Omega} p(\omega) \cdot \int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)} v^\omega(x_1, \ldots, x_n)\mathrm{d}\mu^\omega(x_1, \ldots, x_n).$$

In Appendix A.1, we show that the set $\mathcal{F}_p$ can be expressed through feasible transportation plans $\mathcal{M}$ with admissible marginals as follows:

$$\mathcal{F}_p = \bigcup_{\substack{\text{asmissible marginals} \\ (\lambda_i^\omega)_{i\in N, \omega\in\Omega} \subset \Delta(\Delta(\Omega))}} \prod_{\omega\in\Omega} \mathcal{M}\big(\lambda_1^\omega, \ldots, \lambda_n^\omega\big). \tag{5}$$

In other words, conditional distributions are feasible if they have admissible marginals. This representation allows us to conduct the maximization in two steps — first, over transportation plans and then over admissible marginals — and leads to the desired formula (4).

Characterization of feasible distributions similar to (5) have appeared in the literature, in particular cases such as binary states Gutmann et al. (1991); Dawid et al. (1995); Burdzy and Pitman (2020); Arieli et al. (2021a) and finite sets of signals Levy et al. (2020); Herings et al. (2020). For reader's convenience, we include a proof in the appendix and also check there that $\mathcal{F}_p$ is a closed convex subset of $\Big(\Delta\big(\Delta(\Omega) \times \ldots \times \Delta(\Omega)\big)\Big)^\Omega$; see Lemma 5.

It is instructive to compare the characterization of feasible conditional distributions of beliefs (5) to the characterizations of unconditional ones found by Dawid et al. (1995) for two receivers in the binary-state case and by Arieli et al. (2021a) for any number of receivers. In our notation, the set they characterize can be seen as the image of $\mathcal{F}_p$ under the linear map $(\mu^\omega)_{\omega\in\Omega} \to \sum_{\omega\in\Omega} p(\omega) \cdot \mu^\omega$. This image does not admit a simple characterization in terms of marginals; in particular, the conditions found Dawid et al. (1995) and Arieli et al. (2021a) are rather involved. The surprising simplicity of characterization (5) underlies the connection to optimal transport and drives our analysis.

Below we consider several classes of problems that can be solved explicitly using Theorem 1.

3.1 One-state persuasion

A problem B is a *one-state persuasion problem* if the utility function v^ω has the following form

$$v^\omega(x_1,\ldots,x_n) = \begin{cases} v(x_1,\ldots,x_n), & \omega = \omega_0 \\ 0, & \omega \neq \omega_0 \end{cases},$$

where $\omega_0 \in \Omega$ is fixed and v is some measurable function $\Delta(\Omega)^N \to \mathbb{R}$.

Remark 1. One-state persuasion problems arise naturally if at each state ω, the sender derives utility from disjoint groups $N_\omega \subset N$ of receivers, e.g., a PR-manager targets different parts of the population depending on the focus ω of a PR-campaign. Formally, consider a partition $N = \bigcup_{\omega\in\Omega} N_\omega$ of the set of receivers into $|\Omega|$ disjoint subsets and assume that the sender's utility takes the following form $v = v^\omega\big((x_i)_{i\in N_\omega}\big)$. Such a persuasion problem boils down to solving $|\Omega|$ one-state persuasion problems indexed by $\omega_0 \in \Omega$ and having N_{ω_0} as the set of receivers and the utility equal to $v^{\omega_0}\big((x_i)_{i\in N_{\omega_0}}\big)$ if $\omega = \omega_0$ and to zero, otherwise.

For one-state problems, only the state ω_0 contributes to the formula for the value in Theorem 1. In particular, the only components of the admissible marginals playing a role are $(\lambda_i^{\omega_0})_{i\in N}$.

Lemma 1. *For a collection of distributions $(\gamma_i)_{i\in N} \subset \Delta(\Delta(\Omega))$ one can find admissible marginals $(\lambda_i^\omega)_{i\in N,\omega\in\Omega}$ such that $\gamma_i = \lambda_i^{\omega_0}$, $i \in N$, if and only if the following family of inequalities hold:*

$$\int_{\Delta(\Omega)} \frac{x(\omega)}{x(\omega_0)} \mathrm{d}\gamma_i(x) \leqslant \frac{p(\omega)}{p(\omega_0)}, \qquad i \in N,\ \omega \in \Omega\backslash\{\omega_0\}. \tag{6}$$

The lemma is proved in the appendix. The necessity of conditions (6) is easy to see informally. The definition of admissibility (3) implies that there exists a probability measure λ_i such that $\mathrm{d}\lambda_i(x) = \frac{p(\omega)}{x(\omega)}\mathrm{d}\lambda_i^\omega(x)$ (except for points where $x(\omega) = 0$). Hence, $\frac{p(\omega_0)}{x(\omega_0)}\mathrm{d}\lambda_i^{\omega_0}(x) = \frac{p(\omega)}{x(\omega)}\mathrm{d}\lambda_i^\omega(x)$ or, equivalently, $\frac{x(\omega)}{x(\omega_0)}\mathrm{d}\lambda_i^{\omega_0}(x) = \frac{p(\omega)}{p(\omega_0)}\mathrm{d}\lambda_i^\omega(x)$. Integrating this identity over $x \in \Delta(\Omega)$ with $x(\omega_0) \neq 0$, we get (6).

From Theorem 1 and Lemma 1, we conclude that the value of the persuasion problem can be represented as $p(\omega_0)\cdot\sup_\gamma \int v(x)\mathrm{d}\gamma(x)$, where the supremum is over distributions $\gamma \in \Delta\big(\Delta(\Omega)\times\ldots\times\Delta(\Omega)\big)$ such that its marginals $(\gamma_i)_{i\in N}$ satisfy the inequalities (6). We maximize a linear functional over a convex set and hence, by Bauer's principle, we can restrict the maximization to extreme points of this set. Extreme γ turn out to have a simple form. Indeed, extreme points of the set of all probability measures are just point masses (the Dirac delta-measures). The set of feasible γ is cut from the set of all probability measures by $|N|\cdot(|\Omega|-1)$ linear inequalities and, hence, the extreme γ are convex combinations of at most $|N|\cdot(|\Omega|-1)+1$ point masses. The following lemma formalizes this observation.

Lemma 2. *The value of a one-state persuasion problem B can be expressed as the supremum over distributions γ supported on at most $|N| \cdot (|\Omega| - 1) + 1$ points:*

$$\mathrm{Val}[B] = p(\omega_0) \cdot \sup_{\substack{\gamma \in \Delta(\Delta(\Omega) \times \ldots \times \Delta(\Omega)) \\ \text{such that the marginals satisfy (6) and} \\ |\mathrm{supp}\,[\gamma]| \leqslant |N| \cdot (|\Omega| - 1) + 1}} \int_{\Delta(\Omega) \times \ldots \times \Delta(\Omega)} v(x_1, \ldots, x_n)\, \mathrm{d}\gamma(x_1, \ldots, x_n). \tag{7}$$

Note that for γ from the lemma, the integral in (7) as well as the integrals in (6) are, in fact, finite sums with at most $N(|\Omega| - 1) + 1$ summands. In Appendix A, we prove a strengthening of Lemma 2 with the bound on the number of atoms depending on the number of "active" constraints (6); we also demonstrate there that the sender can achieve the utility level corresponding to a distribution γ by using an information structure with at most $|N| \cdot (|\Omega| - 1)$ signals per receiver.

The possibility to reduce one-state persuasion to a finite-dimensional problem can be seen as a peculiar geometric property of the set $\mathcal{F}_p$ of feasible conditional distributions of beliefs. The set of distributions with marginals satisfying (6) can be seen as the image of $\mathcal{F}_p$ under the projection $(\mu^\omega)_{\omega \in \Omega} \to \mu^{\omega_0}$. The fact that this image has extreme points with finite support and a simple structure is to be contrasted with the complicated structure of extreme points of the set $\mathcal{F}_p$ itself. Indeed, Arieli et al. (2021a) showed that feasible unconditional distributions of beliefs (i.e., the image of $\mathcal{F}_p$ under $(\mu^\omega)_{\omega \in \Omega} \to \sum_{\omega \in \Omega} p(\omega) \cdot \mu^\omega$) have extreme points with infinite support, which implies the existence of infinitely-supported extreme points in $\mathcal{F}_p$ since an extreme point of the image under a linear map is the image of an extreme point.

Example 1. To illustrate how to use Lemma 2, consider a persuasion problem where the sender wants to maximize the discord between two receivers in one of the states. The state is binary, $\omega \in \Omega = \{\ell, h\}$, and we identify $x \in \Delta(\Omega)$ with $x(\ell) \in [0, 1]$. Consider the utility function that equals $\left(x_1 - x_2\right)^2$ if the state is $\omega = \ell$, and equals 0 if the state is $\omega = h$, and assume that the prior $p = \frac{1}{2}$.

For two receivers, it is enough to consider distributions γ in (7) with at most three points in the support. If we restrict the maximization to one-point distributions, then the optimum of $\frac{1}{8}$ is achieved at the point mass at a pair of beliefs $(x_1, x_2) = \left(1, \frac{1}{2}\right)$ and also at the pair of beliefs $(x_1, x_2) = \left(\frac{1}{2}, 1\right)$. For γ supported on two points, we can improve the value of the objective to $\frac{2}{9}$, which is achieved at the distribution that places equal weight on $\left(1, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, 1\right)$. Allowing for the third point in the support does not improve the objective.

We conclude that the value of the persuasion problem equals $\frac{2}{9}$. This value is attained at the following information structure I with two signals L, H for both receivers. In state $\omega = \ell$, the sender picks a receiver uniformly at random and reveals her the state (i.e., sends the signal L). In all other cases (state $\omega = \ell$ and the receiver not picked or state $\omega = h$) the sender sends the signal H.

3.2 The case of two receivers and a binary state

Consider a persuasion problem B with two receivers $N = \{1, 2\}$ and a binary state $\omega \in \Omega = \{\ell, h\}$. Similarly to Example 1, we identify $x \in \Delta(\Omega)$ with $x(\ell) \in [0, 1]$. Accordingly, $\Delta(\Omega)$ is identified with the interval $[0, 1]$ and $\Delta(\Delta(\Omega))$ with the set of distributions over it.

We exemplify what the optimization problem from Theorem 1 looks like in this case. The condition of admissibility for a family of distributions λ_1^ℓ, λ_1^h, λ_2^ℓ, λ_2^h on $[0, 1]$ is expressed as $\frac{\mathrm{d}\lambda_i^\ell}{\mathrm{d}\lambda_i}(x) = \frac{x}{p}$ and $\frac{\mathrm{d}\lambda_i^h}{\mathrm{d}\lambda_i}(x) = \frac{1-x}{1-p}$. Excluding λ_i, we see that the admissibility is equivalent to the identity[3] $p(1-x)\,\mathrm{d}\lambda_i^\ell(x) = (1-p)\cdot x\,\mathrm{d}\lambda_i^h(x)$. Hence, the formula for the value (4) reads as follows:

$$\mathrm{Val}[B] = \sup_{\substack{\lambda_1^\ell, \lambda_1^h, \lambda_2^\ell, \lambda_2^h \in \Delta([0,1]) \\ \text{such that} \\ p(1-x)\,\mathrm{d}\lambda_i^\ell(x) = (1-p)\cdot x\,\mathrm{d}\lambda_i^h(x)}} \left(p \cdot \max_{\substack{\mu^\ell \in \Delta([0,1]^2) \\ \text{with marginals } \lambda_1^\ell, \lambda_2^\ell}} \int_{[0,1]^2} v^\ell(x_1, x_2)\,\mathrm{d}\mu^\ell(x_1, x_2) + \right.$$
$$\left. + (1-p) \cdot \max_{\substack{\mu^h \in \Delta([0,1]^2) \\ \text{with marginals } \lambda_1^h, \lambda_2^h}} \int_{[0,1]^2} v^h(x_1, x_2)\,\mathrm{d}\mu^h(x_1, x_2) \right). \tag{8}$$

Solving the optimization problem (8) poses two challenges. First, we need to be able to solve the two transportation problems (the maximization of the integrals) for all possible marginals. Second, we need to be able to maximize the outcome over four marginals. To conquer the first challenge, we make structural assumptions on v^ω and use results from the transportation literature. The remaining optimization over four distributions may also be challenging. However, under symmetry assumption, this step reduces to tractable optimization over a single distribution.

We say that a persuasion problem is *agent-symmetric* if $v^\omega(x_1, x_2) = v^\omega(x_2, x_1)$; it is *state-symmetric* if $v^\ell(x_1, x_2) = v^h(1 - x_1, 1 - x_2)$ and $p = \frac{1}{2}$. For agent-symmetric problems, one can assume that $\lambda_1^\omega = \lambda_2^\omega$ and, for state-symmetric, that λ_i^h is obtained from λ_i^ℓ by the reflection around $\frac{1}{2}$; see Lemma 6 in Appendix A. We demonstrate the applicability of Theorem 1 by providing a closed-form solution for symmetric supermodular utilities.

[3] This identity is to be understood in the integrated sense, i.e., $p \cdot \int_{[0,1]} f(x)(1-x)\,\mathrm{d}\lambda_i^\ell(x) = (1-p)\int_{[0,1]} f(x) \cdot x\,\mathrm{d}\lambda_i^h(x)$ for any continuous function f on $[0, 1]$.

3.2.1 Supermodular persuasion

Recall that a function $v : [0,1]^2 \to \mathbb{R}$ is *supermodular* if for all $x_1 \leqslant x_1'$ and $x_2 \leqslant x_2'$

$$v(x_1, x_2) + v(x_1', x_2') \geqslant v(x_1, x_2') + v(x_1', x_2). \tag{9}$$

Informally, the definition of supermodularity (9) requires that, if we are given a pair of points for each of the coordinates, the function is maximized if these pairs are coupled in the monotone way, i.e., when one coordinate is large another is also large. This insight is formalized and generalized in the theory of optimal transportation. For a pair of distributions $\lambda_1, \lambda_2 \in \Delta([0,1])$, their *co-monotone coupling* $\gamma_{\lambda_1 \uparrow\uparrow \lambda_2}$ is defined as the distribution of the vector[4] $(f_{\lambda_1}(\xi), f_{\lambda_2}(\xi))$, where ξ is a random variable with the uniform distribution on $[0,1]$ and f_λ denotes the inverse cumulative distribution functions of a distribution $\lambda \in \Delta([0,1])$, i.e., $f_\lambda(t) = \min\{x \in [0,1] \,:\, \lambda([0,x]) \geqslant t\}$. It is easy to see that $\gamma_{\lambda_1 \uparrow\uparrow \lambda_2}$ has λ_1, λ_2 as the marginals and, hence, belongs to the set of feasible plans for a transportation problem with these marginals. Any transportation problem with a supermodular utility v has the co-monotone coupling as the optimal solution:

$$T_v[\lambda_1, \lambda_2] = \int_{[0,1]^2} v(x_1, x_2) \mathrm{d}\gamma_{\lambda_1 \uparrow\uparrow \lambda_2}(x_1, x_2) = \int_{[0,1]} v\Big(f_1(t), f_2(t)\Big) \mathrm{d}t, \tag{10}$$

see Theorem 3.12 in Rachev and Rüschendorf (1998).[5]

We call a persuasion problem *supermodular* if $N = \{1,2\}$, $\Omega = \{\ell, h\}$, and the utility function $v^\omega(x_1, x_2)$ is a supermodular function in each of the two states. Thanks to (10), the internal transportation problems in equation (8) can be solved explicitly and, hence, it remains to maximize the outcome over the admissible marginals $(\lambda_i^\omega)_{i \in \{1,2\}, \omega \in \{\ell, h\}}$ to compute the value.

The following Lemma shows that supermodular agent-symmetric problems reduce to persuading one auxiliary receiver and, hence, are easy to solve.

Lemma 3. *An agent-symmetric supermodular persuasion problem B is equivalent to a single-receiver persuasion problem B' that has the same prior p and the utility $v'^\omega(x) = v^\omega(x,x)$. Namely,*

$$\mathrm{Val}[B] = \mathrm{Val}\big[B'\big] = \mathrm{cav}\Big[\overline{v'}\Big](p),$$

where $\overline{v'}(x) = x \cdot {v'}^\ell(x) + (1-x){v'}^h(x)$ and $\mathrm{cav}\Big[\overline{v'}\Big]$ denotes the concavification of $\overline{v'}$. Moreover, information structures with two public signals are enough for optimal persuasion.

Lemma 3 is proved in Appendix A. The intuition is as follows. If the problem is agent-symmetric, we can restrict the maximization to admissible marginals satisfying $\lambda_1^\omega = \lambda_2^\omega$. Their co-monotone

[4] This distribution is also known as Fréchet upper bound (Joe, 1997).

[5] The original result is due to Lorentz (1949) and holds in any dimension.

coupling is supported on the diagonal $x_1 = x_2$, which corresponds to information structures with public signals. Multi-receiver persuasion with public signals is equivalent to persuading one representative receiver and we get the result.

We note that Lemma 3 extends to any number of receivers straightforwardly since (10) admits such an extension.

Example 2. Consider a one-state persuasion problem with $v^\ell \equiv 0$ and $v^h(x_1, x_2) = g(x_1)g(x_2)$, which is supermodular for non-decreasing g. The function $\overline{v'}$ from Lemma 3 equals $(1-x)\big(g(x)\big)^2$. For $g(x) = \sqrt{x}$, this function is concave and we conclude that revealing no information is optimal for any prior p. For $g(x) = x$, the function $\overline{v'}$ is convex on $\left[0, \frac{1}{3}\right]$ and concave on $\left[\frac{1}{3}, 1\right]$. Hence, for p in $\left[\frac{1}{3}, 1\right]$, revealing no information is optimal. For $p \in \left[0, \frac{1}{3}\right]$, the concavification of $\overline{v'}$ is given by the linear interpolation of its values at 0 and $\frac{1}{3}$, i.e., the value equals $\frac{2}{9}p$; the optimal information structure induces the posterior beliefs $x_1 = x_2$ equal to either 0 or $\frac{1}{3}$, e.g., $S_1 = S_2 = \{L, H\}$ and $s_1 = s_2 = L$ is always sent to both receivers in the low state $\omega = \ell$, while, in the high state, the sender randomizes between L and H with probabilities $\frac{2p}{1-p}$ and $\frac{1-3p}{1-p}$.

4 Analog of Kantorovich–Rubinstein duality for persuasion

One of the main tools in optimal transportation is the dual representation for the optimal value, the so-called Kantorovich-Rubinstein duality, which we discuss below in detail. Using this classic result as an inspiration, we derive a dual representation for the value of a persuasion problem. We compare our formula to the Kantorovich-Rubinstein duality and to the duality described by Dworczak and Kolotilin (2019) for persuasion with one receiver. As an application of the dual representation, we find a multi-receiver extension of the celebrated result that the value of a single-receiver persuasion problem coincides with the concavification of the utility function. We also show how one can construct an explicit solution to the dual problem and use it to solve the primal problem.

The following theorem is the main result of this section.

Theorem 2. *Consider a persuasion problem B with an upper semi-continuous utility function. The value of B can be represented as follows:*

$$\mathrm{Val}[B] = \inf_{\substack{V^\omega \in \mathbb{R},\text{ continuous } \varphi_i^\omega \text{ on } \Delta(\Omega) \text{ such that} \\ v^\omega(x_1,\ldots,x_n) \leqslant V^\omega + \sum_{i\in N} \varphi_i^\omega(x_i) \\ \text{and } \sum_{\omega\in\Omega} x_i(\omega)\varphi_i^\omega(x_i) = 0}} \sum_{\omega\in\Omega} p(\omega) V^\omega. \tag{11}$$

If the utility function is continuous, the optimum is attained, i.e., the infimum can be replaced by the minimum.

The theorem has a simple geometric interpretation. Recall that the support function of a convex set $\mathcal{V} \subset \mathbb{R}^d$ is a convex function defined by $h_{\mathcal{V}}(t) = \sup_{V \in \mathcal{V}} \langle t, V \rangle$, $t \in \mathbb{R}^d$. From (11), we see that the value coincides with $-h_{\mathcal{V}}\big(-p\big)$, where $\mathcal{V} = \big\{(V^\omega)_{\omega \in \Omega} : \exists \varphi_i^\omega \text{ satisfying the constraints}\big\}$. This set is convex[6] and does not depend on the prior p. In particular, the value is a convex function of the prior.

Comparison to the Kantorovich-Rubinstein duality. Kantorovich and Rubinstein found the dual to the transportation problem in the case of two marginals. The multi-marginal version of their result takes the following form

$$T_v\big[(\lambda_i)_{i \in N}\big] = \inf_{\substack{V \in \mathbb{R},\ \text{continuous } \varphi_i :\ X_i \to \mathbb{R} \\ \text{such that } v(x_1,\ldots,x_n) \leqslant V + \sum_{i \in N} \varphi_i(x_i) \\ \text{and } \int_{X_i} \varphi_i(x_i) \mathrm{d}\lambda_i(x_i) = 0}} V, \tag{12}$$

where X_i, $i \in N$, are compact metric spaces and v is an upper semicontinuous function on their Cartesian product; the optimum exist provided that v is continuous (Rachev and Rüschendorf, 1998).

The similarity between formula (11) from Theorem 2 and the Kantorovich-Rubinstein duality is not surprising thanks to the connection between persuasion and transportation established in Theorem 1. The differences are caused by the fact that the marginals in Theorem 1 are not fixed but instead are free parameters that satisfy the admissibility constraints. Hence, in contrast to (12), the marginals do not enter (11) and the functions φ_i^ω are required to satisfy the pointwise orthogonality requirement $\sum_{\omega \in \Omega} x_i(\omega) \varphi_i^\omega(x_i) = 0$ instead of functional orthogonality to measures λ_i as in (12).

A version of the Kantorovich-Rubinstein duality persists for general bounded measurable utilities v with general measurable φ_i (Kellerer, 1984, Theorem 2.14). We expect that Theorem 2 also admits such an extension.

Comparison to the single-receiver case. Consider a persuasion problem with one receiver and the utility function $v^\omega = v$ independent of the state. Dworczak and Kolotilin (2019) established[7] a

[6] If $\big(V^\omega, \varphi_i^\omega\big)$ and $\big(V'^\omega, \varphi_i'\big)$ both satisfy the constraints so does their convex combination and, hence, $\mathcal{V}$ is convex.

[7] For a finite dimension ($|\Omega| < \infty$), the result is intuitive. The value is known to be equal to the concavification $\mathrm{cav}[v](p)$ and the concavification of a function is the envelope of affine functions that lie above it. Dworczak and Kolotilin (2019) demonstrated that this remains true in the far less intuitive infinite-dimensional case, e.g., for continuous Ω.

dual representation for the value, which, in our notation, can be written as follows:

$$\mathrm{Val}[B] = \inf_{\substack{V^\omega \in \mathbb{R} \text{ such that} \\ v(x) \leqslant \sum_{\omega\in\Omega} x(\omega)\cdot V^\omega}} \sum_{\omega\in\Omega} p(\omega)\cdot V^\omega. \tag{13}$$

The crucial difference between (13) and Theorem 2 is that functions φ_i^ω are absent in the single-receiver case. As a consequence, the problem with one receiver is finite-dimensional, while that from Theorem 2, infinite-dimensional.

One may wonder if we can assume that $\varphi_i^\omega \equiv 0$ in Theorem 2. For more than one receiver, the answer is negative even if the utility function is state-independent and satisfies all the symmetries. Below we will see an example with two receivers, where the optimum is attained at non-linear functions φ_i^ω. We believe that, as in the theory of optimal transportation, the minimization cannot be restricted to functions φ_i^ω having a simple parametric form. This can be seen as another justification for the difficulty of multi-receiver persuasion.

Note that Theorem 2 and (12) are examples of infinite-dimensional programs, where the existence of the optimum is guaranteed under a simple condition of continuity. By contrast, when (13) becomes infinite-dimensional (for infinite Ω), the existence of the optimum requires superdifferentiability of the concavified utility function, a hard-to-check condition.

Proof idea of Theorem 2. The differences between Theorem 2 and the Kantorovich-Rubinstein duality do not allow us to deduce the former from the latter. In Appendix B.1, we use a game-theoretic approach to derive the dual. We define an auxiliary zero-sum game with a sup-inf value equal to the value of the persuasion problem, use Sion's minimax theorem to exchange sup and inf, and show that the inf-sup value coincides with the right-hand side of (11).

Let $\|v\|_\infty$ be the maximal absolute value of v. To prove the existence of the optimum for continuous v, we show that one can restrict the minimization to $|V^\omega| \leqslant \frac{2}{p(\omega)}\|v\|_\infty$ and φ_i^ω bounded in absolute value by $\frac{2n}{p(\omega)}\|v\|_\infty$ and having moduli of continuity upper-bounded in terms of the modulus of v (Lemma 7 in the appendix). The existence of the optimum then follows from compactness of this set.

4.1 Analog of the concavification formula for the value

Consider a single-receiver persuasion problem with a continuous state-independent utility function $v(x)$. The value of this problem is equal to the concavification $\mathrm{cav}[v](p)$ (Kamenica and Gentzkow, 2011). Notice that $u = \mathrm{cav}[v]$ is a concave continuous function and, in particular, revealing no information would be optimal if the utility function was equal to $u(x)$. Hence, the classic concavification result can be restated as follows.

Observation 1 (reinterpreted concavification)**.** *For a single-receiver persuasion problem $B = (\Omega, p, v)$ with continuous v, the following identity holds:*

$$\mathrm{Val}[B] = \min_{\substack{\text{continuous } u \text{ such that} \\ v \leqslant u \text{ and} \\ \text{non-revealing is optimal for } (\Omega, q, u)\ \forall q}} u(p). \tag{14}$$

In this form, the result remains valid for any number of receivers and for state-dependent utilities.

Theorem 3. *For a persuasion problem $B = (\Omega, p, N, v)$ with an upper semicontinuous v, the following identity holds:*

$$\mathrm{Val}[B] = \inf_{\substack{\text{continuous } u \text{ such that} \\ v^\omega(x_1, \ldots, x_n) \leqslant u^\omega(x_1, \ldots, x_n) \text{ and} \\ \text{non-revealing is optimal for } (\Omega, q, N, u)\ \forall q}} \sum_{\omega \in \Omega} p(\omega) \cdot u^\omega(p, p, \ldots, p). \tag{15}$$

For continuous v, the optimum is achieved and inf *can be replaced by* min.

The proof of Theorem 3 relies on duality (Theorem 2) and is relegated to Appendix B.2.

Theorem 3 relates two seemingly incomparable problems: (a) Compute the value of an arbitrary persuasion problem, and (b) Determine whether the policy of revealing no information is optimal. One might think that problem (b) is significantly simpler. However, Theorem 3 indicates that if one knows how to solve problem (b), all that remains is to minimize over the functions that satisfy (b). As it is believed that (a) is a complicated problem, this indicates that so is problem (b).

Another interesting aspect of Theorem 3 is the central role played by the values of u^ω at the diagonal. Indeed, we evaluate u^ω at the point $(p, p, \ldots, p)$ only although u^ω is a multidimensional function. The informal reason why this "local" information turns out to be enough to characterize the value is the presence of the "global" condition that no-information is optimal.

4.2 Solving the dual problem

We start by illustrating Theorem 2 in the case of two receivers and a binary state. As previously, we identify the set of beliefs $\Delta(\Omega)$ with the interval $[0, 1]$.

The unique feature of the binary-state case is that the last condition in (11) uniquely determines φ^h for a given φ^ℓ. This allows us to simplify the problem by optimizing over two functions instead of four. Denote $\frac{\varphi_i^\ell(x)}{1-x}$ by $\alpha_i(x)$; hence, $\varphi_i^h(x) = -x \cdot \alpha_i(x)$ and we see that α_i is not singular at

$x = 1$. Therefore, (11) reduces to

$$\mathrm{Val}[B] = \inf_{\substack{V^\omega \in \mathbb{R}, \quad \text{continuous } \alpha_i \text{ on } [0,1] \text{ such that} \\ v^\ell(x_1,x_2) \quad \leqslant V^\ell + (1-x_1)\alpha_1(x_1) + (1-x_2)\alpha_2(x_2) \\ v^h(x_1,x_2) \quad \leqslant V^h - x_1 \cdot \alpha_1(x_1) - x_2 \cdot \alpha_2(x_2)}} p \cdot V^\ell + (1-p)V^h. \tag{16}$$

If the problem is symmetric, (16) can be simplified further. For agent-symmetric problems ($v^\omega(x_1, x_2) = v^\omega(x_2, x_1)$), the minimization can be restricted to $\alpha_1 = \alpha_2$; for state-symmetric problems ($p = \frac{1}{2}$ and $v^\ell(x_1, x_2) = v^h(1-x_1, 1-x_2)$) one can assume[8] $\alpha_i(x) = -\alpha_i(1-x)$ and to check only one of the two inequality conditions in (16).

4.2.1 When full-information/partial-information policy is optimal?

As an application of Theorem 2, we will describe a family of persuasion problems in which revealing the state to one receiver and partially informing the other is optimal. We refer to such a signalling policy as a *full-information/partial-information policy.*

The idea is that, in problems where one receiver gets fully informed, the optimal α_i and V^ω are determined by the values of the utility function on the boundary of $[0,1]^2$. Relying on this intuition, we construct a candidate solution to the dual problem. The requirement that the constructed candidate solution is, in fact, a solution will pin down the class of persuasion problems where full-information/partial-information policy is optimal.

Consider an agent-symmetric persuasion problem B with continuous utility v. By symmetry, we can assume $\alpha_1 = \alpha_2 = \alpha$ in (16). Let us describe the set of V^ℓ and V^h such that the pair of inequalities from (16)

$$\begin{aligned} v^\ell(x_1,x_2) &\leqslant V^\ell + (1-x_1)\alpha(x_1) + (1-x_2)\alpha(x_2), \\ v^h(x_1,x_2) &\leqslant V^h - x_1 \cdot \alpha(x_1) - x_2 \cdot \alpha(x_2) \end{aligned} \tag{17}$$

has a solution and construct such a solution α. Plugging $x_2 = 1$ to the first inequality and $x_2 = 0$ to the second, we see that

$$\frac{v^\ell(x_1,1) - V^\ell}{1-x_1} \leqslant \alpha(x_1) \leqslant \frac{V^h - v^h(x_1,0)}{x_1}. \tag{18}$$

Let us assume for a moment that this inequality is equivalent to the pair (17), i.e., we assume that the inequality in the interior of the square follows from the inequality on the boundary. In the end,

[8] For an agent-symmetric problem and distinct α_1, α_2 satisfying the constraints for some V^ω, define $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \frac{\alpha_1+\alpha_2}{2}$. The functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ satisfy the constraints with the same $\tilde{V}^\omega = V^\omega$ and, hence, give the same value to the objective. For state-symmetric problems, the argument is analogous with $\tilde{\alpha}_i(x) = \frac{\alpha_i(x)+\alpha_i(1-x)}{2}$ and $\tilde{V}^\omega = \frac{V^\ell+V^h}{2}$.

this assumption will pin down the class of functions v for which the candidate solution will indeed be a solution.

To optimize the objective in (16), we want to make both V^ℓ and V^h as small as possible. However, the inequality (18) provides a natural obstacle for making both numbers too low: the lower bound cannot exceed the upper bound. Hence the optimal choices correspond to pairs of V^ℓ and V^h such that the graph of the lower bound in (18) touches the graph of the upper bound. Equivalently, the linear segment $x_1 \cdot V^\ell + (1-x_1)V^h$ touches from above the graph of

$$\overline{v}(x_1) = x_1 \cdot v^\ell(x_1, 1) + (1-x_1)v^h(x_1, 0).$$

By (16), the value of the persuasion problem is the minimal value of the expression $p \cdot V^\ell + (1-p)V^h$. We conclude that $\mathrm{Val}[B] = \mathrm{cav}[\overline{v}](p)$ if (17) and (18) are indeed equivalent, but, in general, $\mathrm{cav}[\overline{v}](p)$ is the lower bound on $\mathrm{Val}[B]$ since the inequalities (17) may exclude some pairs (V^ℓ, V^h) thus increasing the minimum.

Let us find a sufficient condition for the equivalence of (17) and (18). For this purpose, we describe a reasonable candidate choice of a function $\alpha = \alpha_p$ satisfying (18). Let V_p^ℓ and V_p^h be numbers such that

$$x_1 \cdot V_p^\ell + (1-x_1)V_p^h \quad \text{is the tangent line to the graph of} \quad \mathrm{cav}[\overline{v}] \quad \text{at } x_1 = p. \tag{19}$$

Note that if $\mathrm{cav}[\overline{v}]$ is differentiable at $x_1 = p$, then $V_p^\ell = \mathrm{cav}[\overline{v}](p) + (1-p)\frac{d}{dx_1}\mathrm{cav}[\overline{v}](p)$ and $V_p^h = \mathrm{cav}[\overline{v}](p) - p\frac{d}{dx_1}\mathrm{cav}[\overline{v}](p)$.

Let b_p be the leftmost point where this tangent line touches the graph, and let c_p be the rightmost such point. We define α_p as follows:

$$\alpha_p(x_1) = \begin{cases} \frac{v^\ell(x_1,1) - V_p^\ell}{1-x_1}, & x_1 \leqslant b_p \\ v^\ell(x_1, 1) - V_p^\ell + V_p^h - v^h(x_1, 0), & x_1 \in [b_p, c_p] \\ \frac{V_p^h - v^h(x_1,0)}{x_1}, & x_1 \geqslant c_p \end{cases}. \tag{20}$$

In other words, for small values of x_1, the function α_p is given by the lower bound in (18), for high values x_1 it is given by the upper bound, and between the points b_p and c_p (at these points the two bounds coincide) α_p equals the convex combination of the two bounds with weights $(1-x_1)$ and x_1 correspondingly. The intuition is that α_p must be equal to the "most demanding" of the bounds, e.g., for small x_1, the upper bound is unlikely to be active thanks to x_1 in the denominator.

Lemma 4. *If α_p, V_p^ℓ, and V_p^h defined by (20) and (19) satisfy the inequalities (17), then*

$$\mathrm{Val}[B] = \mathrm{cav}[\overline{v}](p)$$

and an optimal information structure reveals the state to receiver 2 and induces the posterior belief x_1 of receiver 1 equal to either b_p or c_p.

Proof. As we argued above, $\mathrm{cav}[\overline{v}](p)$ is the lower bound on the value. Alternatively, one can see it directly by using the information structure from the statement of the the lemma, which gives $\mathrm{cav}[\overline{v}](p)$ to the sender. It remains to show that the sender cannot improve upon this utility level. Substituting α_p, V_p^ℓ, and V_p^h into the dual representation for the value (16), we see that $\mathrm{Val}[B] \leqslant p \cdot V_p^\ell + (1-p)V^h = \mathrm{cav}[\overline{v}](p)$, which completes the proof. □

Whenever v is given explicitly, checking the condition of Lemma 4 boils down to checking the inequalities between explicitly given functions on the unit square.

Example 3. The state-independent persuasion problem with the utility function $v^\ell(x_1, x_2) = v^h(x_1, x_2) = |x_1 - x_2|$ satisfies the condition for any prior $p \in (0, 1)$. This problem was considered by Arieli et al. (2021a) for $p = \frac{1}{2}$ and by Burdzy and Pitman (2020) for all priors. We obtain a simple alternative proof that the value of this problem equals $2p(1-p)$ and the policy revealing the state to one of the receivers is optimal ($b_p = c_p = p$ for this problem).

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A Proofs for Section 3

A.1 Proof of Theorem 1

Consider a persuasion problem $B = (\Omega, p, N, v)$. By the definition, its value $\mathrm{Val}[B]$ is the maximal expected utility $\sup_I \mathbb{E}_I[v^\omega(x_1, \ldots, x_n)]$, where the maximization is over all information structures. To compute the expectation, we only need to know the joint distribution of $(\omega, x_1, \ldots, x_n)$ and we know that the marginal of this distribution on ω equals p. Hence, to reconstruct the whole distribution, it is enough to have the conditional distributions of $(x_1, \ldots, x_n)$ for each realization of $\omega \in \Omega$.

Recall that $(\mu^\omega)_{\omega\in\Omega} \subset \Delta\big(\Delta(\Omega) \times \ldots \times \Delta(\Omega)\big)$ are *feasible conditional distributions of posterior beliefs* if there exists an information structure I such that the joint distribution of posterior beliefs $x_1, \ldots, x_n$ given the state ω equals μ^ω for each $\omega \in \Omega$. By $\mathcal{F}_p$, we denote the set of all such collections $(\mu^\omega)_{\omega\in\Omega}$.

Rewriting the expectation as the integral over the joint distribution we see that

$$\mathrm{Val}[B] = \sup_{(\mu^\omega)_{\omega\in\Omega}\in\mathcal{F}_p} \sum_{\omega\in\Omega} p(\omega) \cdot \int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)} v^\omega(x_1, \ldots, x_n) \mathrm{d}\mu^\omega(x_1, \ldots, x_n). \tag{21}$$

For the next step, we need to characterize the set of feasible distributions $\mathcal{F}_p$. Recall that $\Delta_p(\Delta(\Omega))$ denotes the set of distributions over $\Delta(\Omega)$ with mean p and we call $(\lambda_i^\omega)_{i\in N,\omega\in\Omega} \subset \Delta(\Delta(\Omega))$ are *admissible marginals* if there exist $(\lambda_i)_{i\in N} \subset \Delta_p(\Delta(\Omega))$ such that $\frac{\mathrm{d}\lambda_i^\omega}{\mathrm{d}\lambda_i}(x) = \frac{x(\omega)}{p(\omega)}$.

Lemma 5. *Distributions $(\mu^\omega)_{\omega\in\Omega}$ are feasible conditional distributions of posterior beliefs if and only if they have admissible marginals. The set of all feasible distributions $\mathcal{F}_p$ is a convex subset of $\Delta\big(\Delta(\Omega) \times \ldots \times \Delta(\Omega)\big)$ closed in the topology of weak convergence.*

Before proving the lemma, we check that the theorem is its corollary.

By Lemma 5, we can split the maximization in (21) into two steps, namely, the maximization over admissible marginals and the maximization over arbitrary joint distributions with given marginals. We get

$$\mathrm{Val}[B] = \sup_{\substack{\text{asmissible marginals} \\ (\lambda_i^\omega)_{i\in N,\omega\in\Omega} \subset \Delta(\Delta(\Omega))}} \sum_{\omega\in\Omega} p(\omega) \cdot \sup_{\mu^\omega \in \mathcal{M}\left(\lambda_1^\omega,\ldots,\lambda_n^\omega\right)} \int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)} v^\omega(x_1,\ldots,x_n)\mathrm{d}\mu^\omega(x_1,\ldots,x_n), \tag{22}$$

where $\mathcal{M}\big(\lambda_1^\omega,\ldots,\lambda_n^\omega\big)$ denotes the subset of distributions from $\Delta\big(\Delta(\Omega)\times\ldots\times\Delta(\Omega)\big)$ with marginals $\lambda_1^\omega,\ldots,\lambda_n^\omega$, i.e., the set of feasible transportation plans in the transportation problem $T_{v^\omega}\big[(\lambda_i^\omega)_{i\in N}\big]$. Thus (22) can be rewritten in terms of the transportation problems:

$$\mathrm{Val}[B] = \sup_{\substack{\text{asmissible marginals} \\ (\lambda_i^\omega)_{i\in N,\omega\in\Omega} \subset \Delta(\Delta(\Omega))}} \sum_{\omega\in\Omega} p(\omega) \cdot T_{v^\omega}\big[(\lambda_i^\omega)_{i\in N}\big],$$

which coincides with the formula from the theorem.

Now let us demonstrate the existence of the optimum for upper semicontinuous v. The integral of an upper semicontinuous function over a compact set is an upper semicontinuous function of the distribution in the weak topology (Villani, 2008, Lemma 4.3). Hence, the objective in (21) is upper semicontinuous. An upper semicontinuous function on a compact set attains its maximum. By Lemma 5, the set $\mathcal{F}_p$ is a closed subset of $\Delta\big(\Delta(\Omega)\times\ldots\times\Delta(\Omega)\big)$ and, hence, is compact since the set of all probability distributions over a compact set is compact in the weak topology. We conclude that the maximum over $\mathcal{F}_p$ is attained, i.e., both the optimal marginals and the optimal transportation plans exist.

It remains to prove Lemma 5.

Proof of Lemma 5. First, we show that all feasible distributions have admissible marginals. Fix an information structure I inducing the conditional distributions $(\mu^\omega)_{\omega\in\Omega}$ of posterior beliefs. Pick a receiver $i \in N$ and consider the conditional distributions $(\mu_i^\omega)_{\omega\in\Omega}$ of i's belief x_i (these are marginals of $(\mu^\omega)_{\omega\in\Omega}$ on i's coordinate). We need to demonstrate that there exists λ_i with mean p such that

$$\frac{\mathrm{d}\mu_i^\omega}{\mathrm{d}\lambda_i}(x) = \frac{x(\omega)}{p(\omega)} \tag{23}$$

for all $x \in \Delta(\Omega)$. We will show that this identity holds if λ_i is set to be equal to the unconditional distribution $\mu_i = \sum_{\omega\in\Omega} p(\omega)\cdot\mu_i^\omega$ of i's beliefs.

By the definition of the posterior belief, the conditional distribution of the state ω given that i's belief x_i equals x coincides with x, i.e.,

$$\mathbb{P}_I(\omega = w \mid x_i = x) = x(w) \quad \text{for all } w \in \Omega \text{ and } \mathbb{P}_I\text{-almost-all } x.$$

By the Bayes formula, the left-hand side of this identity rewrites as follows:

$$\mathbb{P}_I(\omega = w \mid x_i = x) = p(w) \cdot \frac{\mathrm{d}\mu_i^\omega}{\mathrm{d}\mu_i}(x).$$

We see that the identity (23) holds. It remains to show that λ_i has the mean p. Indeed, by (23),

$$\int_{\Delta(\Omega)} x(\omega)\,\mathrm{d}\lambda_i(x) = p(\omega) \int_{\Delta(\Omega)} 1\,\mathrm{d}\mu_i^\omega(x) = p(\omega).$$

We conclude that feasible distributions have admissible marginals.

Second, let us demonstrate that any collection $(\mu^\omega)_{\omega\in\Omega}$ with admissible marginals $(\mu_i^\omega)_{i\in N,\omega\in\Omega}$ is feasible. Given such a collection, we construct an information structure $I = ((S_i)_{i\in N}, \pi(\cdot \mid \omega))$ as follows. The sets of signals S_i coincide with $\Delta(\Omega)$ for each receiver i and the distribution of signals $\pi(\cdot \mid \omega)$ at a state ω is equal to μ^ω. In other words, the sender uses μ^ω to generate the collection of signals $(s_1, \ldots, s_n)$ and then tells each receiver i her coordinate s_i. Let us compute the belief x_i induced by the signal s_i using the Bayes formula:

$$x_i(w) = \mathbb{P}_I(\omega = w \mid s_i) = p(\omega) \cdot \frac{\mathrm{d}\mu_i^\omega}{\mathrm{d}\sum_{w'\in\Omega} p(w')\mu_i^{w'}}(s_i). \tag{24}$$

By formula (23), we deduce that λ_i from the admissibility requirement can be expressed as $\lambda_i = \sum_{\omega\in\Omega} p(\omega)\mu_i^\omega$, i.e., λ_i coincides with the distribution in the denominator of (24). Hence, (23) and (24) imply that $x_i = s_i$, i.e., the induced belief coincides with the signal. Thus the joint distribution of beliefs coincides with the joint distribution of signals, i.e., the information structure I induces $(\mu^\omega)_{\omega\in\Omega}$ as the conditional distribution of beliefs. We conclude that any collection of distributions with admissible marginals is feasible.

It remains to check that $\mathcal{F}_p$ is closed convex set. We already know that feasibility is equivalent to admissibility of marginals. Let us rewrite this condition in a form that makes convexity and closedness apparent. We saw that instead of looking for arbitrary λ_i in the admissibility condition, we can check it for $\lambda_i = \sum_{\omega\in\Omega} p(\omega)\mu_i^\omega$. Hence, the admissibility of marginals of $(\mu^\omega)_{\omega\in\Omega}$ can be written as

$$p(\omega)\,\mathrm{d}\mu_i^\omega = x(\omega)\ \mathrm{d}\sum_{\omega'\in\Omega} p(\omega')\mu_i^{\omega'}$$

or, equivalently, in the integrated form:

$$p(\omega) \cdot \int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)} \psi(x_i)\mathrm{d}\mu^\omega(x_1, \ldots, x_n) - \int_{\Delta(\Omega)} x_i(\omega) \cdot \psi(x_i) \left(\sum_{\omega'\in\Omega} p(\omega')\mathrm{d}\mu^{\omega'}(x_1, \ldots, x_n)\right) = 0 \tag{25}$$

for all continuous functions $\psi : \Delta(\Omega) \to \mathbb{R}$. Since, this condition is linear in $(\mu^\omega)_{\omega\in\Omega}$, a convex combination of feasible distributions is also feasible. Since the integrands are continuous functions, the weak limit of a sequence of distributions satisfying the conditions also satisfies them. We get closedness. $\square$

A.2 Proofs for one-state persuasion

Proof of Lemma 1. Let us demonstrate the necessity of the condition (6). In other words, we need to show that if $(\lambda_i^\omega)_{i\in N,\omega\in\Omega}$ are admissible marginals and $\gamma_i = \lambda_i^{\omega_0}$, then

$$\int_{\Delta(\Omega)} \frac{x(\omega)}{x(\omega_0)} \mathrm{d}\gamma_i(x) \leqslant \frac{p(\omega)}{p(\omega_0)}, \qquad i \in N,\ \omega \in \Omega\backslash\{\omega_0\}.$$

By the definition of admissibility, there exists λ_i such that

$$\frac{\mathrm{d}\lambda_i^\omega}{\mathrm{d}\lambda_i}(x) = \frac{x(\omega)}{p(\omega)} \tag{26}$$

for all ω and i. Let $\varepsilon > 0$ be the small parameter. Hence, $\frac{\mathrm{d}\lambda_i^{\omega_0}}{\mathrm{d}\lambda_i}(x) \leqslant \frac{\max\{x(\omega_0),\varepsilon\}}{p(\omega_0)}$ or, equivalently,

$$\frac{1}{\max\{x(\omega_0), \varepsilon\}} \mathrm{d}\lambda_i^{\omega_0}(x) \leqslant \frac{1}{p(\omega_0)} \mathrm{d}\lambda_i(x). \tag{27}$$

By (26), $\frac{x(\omega)}{p(\omega)}\mathrm{d}\lambda_i(x) = \mathrm{d}\lambda_i^\omega(x)$. Applying this identity to (27), we get

$$\frac{x(\omega)}{\max\{x(\omega_0), \varepsilon\}} \mathrm{d}\lambda_i^{\omega_0}(x) \leqslant \frac{p(\omega)}{p(\omega_0)} \mathrm{d}\lambda_i^\omega(x).$$

Integrating this inequality over $\Delta(\Omega)$, we obtain

$$\int_{\Delta(\Omega)} \frac{x(\omega)}{\max\{x(\omega_0), \varepsilon\}} \mathrm{d}\lambda_i^{\omega_0}(x) \leqslant \frac{p(\omega)}{p(\omega_0)}.$$

Letting ε go to zero, gives (6).

Now we check the sufficiency. For given $(\gamma_i)_{i\in N}$ satisfying (6) we need to construct admissible $(\lambda_i^\omega)_{i\in N,\omega\in\Omega}$ such that $\gamma_i = \lambda_i^{\omega_0}$. The idea is to use formula (26) to define λ_i first. Set

$$\mathrm{d}\widetilde{\lambda}_i(x) = \frac{p(\omega_0)}{x(\omega_0)} \mathrm{d}\gamma_i(x).$$

The measure $\widetilde{\lambda}_i$ may not be a probability measure and its mean may not equal p. To make a probability measure with the desired mean out of λ_i, we define λ_i by

$$\lambda_i = \widetilde{\lambda}_i + \sum_{\omega\in\Omega\backslash\{\omega_0\}} \left(p(\omega) - \int_{\Delta(\Omega)} x(\omega)\mathrm{d}\widetilde{\lambda}_i(x) \right) \cdot \delta_\omega, \tag{28}$$

where δ_ω denotes the point mass at ω. By (6), the coefficients in these formula are non-negative and, hence, λ_i is a non-negative measure. By the construction $\int x(\omega)\mathrm{d}\lambda_i = p(\omega)$ and so the mean of λ_i is p. Summing up these equalities, we see that λ_i is a probability measure. For $\omega \neq \omega_0$, define λ_i^ω by (26); λ_i^ω is a probability measure since the mean of λ_i is p. To show that $(\lambda_i^\omega)_{i\in N,\omega\in\Omega}$ with

$\lambda_i^{\omega_0} = \gamma_i$ are admissible marginals, it remains to check that the condition (26) is satisfied at ω_0. Since $x(\omega_0)\mathrm{d}\delta_\omega = 0$ for $\omega \neq \omega_0$, we get $x(\omega_0)\mathrm{d}\lambda_i(x) = x(\omega_0)\mathrm{d}\widetilde{\lambda}_i(x)$. By the definition of $\widetilde{\lambda}_i$,

$$\mathrm{d}\lambda_i^{\omega_0}(x) = \frac{x(\omega_0)}{p(\omega_0)}\mathrm{d}\widetilde{\lambda}_i(x) = \frac{x(\omega_0)}{p(\omega_0)}\mathrm{d}\lambda_i(x).$$

From this identity, we conclude that $\frac{\mathrm{d}\lambda_i^{\omega_0}}{\mathrm{d}\lambda_i}(x) = \frac{x(\omega_0)}{p(\omega_0)}$, which completes the proof of the Lemma 1. $\square$

Proof of Lemma 2: By Lemma 1, the value of a one-state persuasion problem B can be represented as follows:

$$\mathrm{Val}[B] = p(\omega_0) \cdot \sup_{\substack{\gamma \in \Delta(\Delta(\Omega) \times \ldots \times \Delta(\Omega)) \\ \text{such that the marginals satisfy (6)}}} \int_{\Delta(\Omega) \times \ldots \times \Delta(\Omega)} v(x_1, \ldots, x_n)\, \mathrm{d}\gamma(x_1, \ldots, x_n).$$

Our goal is to check that in this formula it is enough to maximize over atomic γ with a certain bound on the number of atoms in the support.

Let $\mathcal{F}_p^{\omega_0}$ be the set of all distributions γ satisfying the inequalities (6). Since these inequalities are linear, $\mathcal{F}_p^{\omega_0}$ is a convex set. The objective linearly depends on $\gamma \in \mathcal{F}_p^{\omega_0}$ and, hence, by the Bauer principle, it is enough to restrict the maximization to the extreme points of the set $\mathcal{F}_p^{\omega_0}$.

To describe the extreme points of $\mathcal{F}_p^{\omega_0}$ let us discuss how the set of extreme points changes when we intersect a convex set with half-spaces. Let X be a convex set with extreme points $X^* \subset X$ and any H be a half-space. The set of extreme points of $X \cup H$ consists of the union of $X^* \cap H$ and extreme points of $(\partial H \cap X)^*$ that are convex combinations $\alpha x + (1-\alpha)x'$ of $x, x' \in X^*$ satisfying the condition $\alpha x + (1-\alpha)x' \in \partial H$, where ∂H denotes the boundary of H. Applying this observation sequentially, we obtain that for the intersection $X \cup \bigcup_{q=1}^{Q} H_q$ with the family of half-spaces any extreme point x^* is given by a convex combination of at most $k+1$ extreme points of X, where k is the number of H_q such that $x^* \in \partial H_q$.

Applying this general statement to our case, we put $X = \Delta(\Delta(\Omega) \times \ldots \times \Delta(\Omega))$ and define the half-spaces $H_{i,\omega}$, $i \in N$, $\omega \in \Omega \backslash \{\omega_0\}$ as the set of signed measures satisfying inequalities (6) with given i and ω. Since the extreme points of X are the point masses, we conclude that the extreme points of $\mathcal{F}_p^{\omega_0}$ are the atomic measures with at most $|N|(|\Omega|-1)+1$ atoms. Hence, one can restrict the maximization to such measures.

This statement can be strengthened. Let $n_i(\gamma)$ be the number of "active" inequalities for the receiver i, i.e., $n_i(\gamma)$ is the number of those inequalities from (6) with the given i that hold as equalities; denote $n(\gamma) = \sum_{i \in N} n_i(\gamma)$. Hence, the extreme γ have at most $n(\gamma)+1$ points in the support. We conclude that

$$\mathrm{Val}[B] = p(\omega_0) \cdot \sup_{\substack{\gamma \in \Delta(\Delta(\Omega) \times \ldots \times \Delta(\Omega)) \\ \text{such that the marginals satisfy (6) and} \\ |\mathrm{supp}\,[\gamma]| \leqslant n(\gamma)+1}} \int_{\Delta(\Omega) \times \ldots \times \Delta(\Omega)} v(x_1, \ldots, x_n)\, \mathrm{d}\gamma(x_1, \ldots, x_n).$$

Let us now discuss how many signals do we need to generate an extreme $\gamma \in \mathcal{F}_p^{\omega_0}$. Using the construction from the proof of Lemma 1, we obtain admissible marginals $(\lambda_i^\omega)_{i \in N, \omega \in \Omega}$ such that $\lambda_i^{\omega_0} = \gamma_i$. Note that the union of supports of λ_i^ω over $\omega \in \Omega$ may be larger than the support of γ_i since we add $|\Omega| - 1 - n_i(\gamma)$ point masses in (28). Let $(\mu^\omega)_{\omega \in \Omega}$ be a feasible family of distributions with $\mu^{\omega_0} = \gamma$ and marginals $(\lambda_i^\omega)_{i \in N, \omega \in \Omega}$; for example, one can take μ^ω to be the product of its marginals for $\omega \neq \omega_0$. In the proof of Theorem 1, we saw that any feasible family $(\mu^\omega)_{\omega \in \Omega}$ can be induced by an information structure with $|\mathrm{supp}\,[\mu_i]|$, where $\mu_i = \sum_{\omega \in \Omega} p(\omega) \mu_i^\omega$. Thus there exists an information structure inducing γ that uses

$$|\mathrm{supp}\,[\gamma_i]| + |\Omega| - 1 - n_i(\gamma) \leqslant |N|(|\Omega| - 1)$$

signals per receiver. □

A.3 Proofs for supermodular persuasion

In this family of results we consider persuasion problems B with two receivers and a binary state. We start with a lemma demonstrating how symmetries of the utility function v simplify the general transportation representation for the value from Theorem 1. Recall that a problem is agent-symmetric if $v^\omega(x_1, x_2) = v^\omega(x_2, x_1)$ and state-symmetric if $v^\ell(x_1, x_2) = v^h(1 - x_1,\, 1 - x_2)$ and $p = \frac{1}{2}$. We start with formally stating the utilization of symmetry in the simplifications of Theorem 1, or more concretely, in equation (8).

Lemma 6. *If B is agent-symmetric,*

$$\mathrm{Val}[B] = \sup_{\substack{\text{admissible } (\lambda_1^\ell, \lambda_1^h, \lambda_2^\ell, \lambda_2^h) \\ \text{such that } \lambda_1^\omega = \lambda_2^\omega = \lambda^\omega}} p \cdot T_{v^\ell}(\lambda^\ell, \lambda^\ell) + (1-p) T_{v^h}(\lambda^h, \lambda^h). \tag{29}$$

If B is state-symmetric,

$$\mathrm{Val}[B] = \sup_{\substack{\text{admissible } (\lambda_1^\ell, \lambda_1^h, \lambda_2^\ell, \lambda_2^h) \\ \text{such that } \lambda_i^\ell([0,\, x]) = \lambda_i^h([1-x,\, 1])}} T_{v^\ell}(\lambda_1^\ell, \lambda_2^\ell) = \tag{30}$$

$$= \sup_{\lambda_1, \lambda_2 \in \Delta([0,1]) \text{ symmetric around } \frac{1}{2}} T_{v^\ell}(\gamma_{\lambda_1}, \gamma_{\lambda_2}), \tag{31}$$

where γ_λ is the distribution having the density $2x$ with respect to λ.

Proof. By Theorem 1,

$$\mathrm{Val}[B] = \sup_{\textit{admissible } (\lambda_1^\ell, \lambda_1^h, \lambda_2^\ell, \lambda_2^h)} p \cdot T_{v^\ell}(\lambda^\ell, \lambda^\ell) + (1-p) T_{v^h}(\lambda^h, \lambda^h). \tag{32}$$

Consider an agent-symmetric problem and let $(\lambda_i^\omega)_{i \in N, \omega \in \Omega}$ be some admissible marginals and $(\mu^\omega)_{\omega \in \Omega}$, some feasible transportation plans with these marginals. Define $\widetilde{\lambda}_1^\omega = \widetilde{\lambda}_2^\omega = \frac{\lambda_1^\omega + \lambda_2^\omega}{2}$ and $\widetilde{\mu}^\omega = \frac{\mu^\omega + \mu^\omega_{x_1 \leftrightarrow x_2}}{2}$, where $\mu^\omega_{x_1 \leftrightarrow x_2}$ denotes the image of μ^ω under the reflection $(x_1, x_2) \to (x_2, x_1)$. Hence, $(\widetilde{\lambda}_i^\omega)_{i \in N, \omega \in \Omega}$ are admissible and $(\widetilde{\mu}^\omega)_{\omega \in \Omega}$ are feasible transportation plans with the same value of the objective in 32. Thus assuming $\lambda_1^\omega = \lambda_2^\omega$ does not change the optimal value and we get (29).

The argument for state-symmetric problems is similar. Let $(\lambda_i^\omega)_{i \in N, \omega \in \Omega}$ be admissible marginals and $(\mu^\omega)_{\omega \in \Omega}$, feasible transportation plans. For a distribution ν on $[0,1]^2$, denote by $\nu_{0 \leftrightarrow 1}$ its image obtained when each of the coordinates x_i is reflected around $\frac{1}{2}$, i.e., $x_i \to 1 - x_i$. Define $\widetilde{\lambda}_i^\ell = \frac{\lambda_i^\ell + \lambda^h_{i,0 \leftrightarrow 1}}{2}$ and $\widetilde{\lambda}_i^h = \frac{\lambda^\ell_{i,0 \leftrightarrow 1} + \lambda_i^h}{2}$ and the transportation plans $\widetilde{\mu}^\ell = \frac{\mu^\ell + \mu^h_{0 \leftrightarrow 1}}{2}$ and $\widetilde{\mu}^h = \frac{\mu^\ell_{0 \leftrightarrow 1} + \mu^h}{2}$. By the construction $\widetilde{\lambda}_i^\ell([0,x]) = \widetilde{\lambda}_i^h([1-x,1])$. The new marginals are admissible and the new transportation plans are feasible and give the same value to the objective in (32). Since both states contribute equally, the objective can be expressed as the doubled contribution of $\omega = \ell$. We get (30).

To derive (31), note that $(\lambda_i^\omega)_{i \in N, \omega \in \Omega}$ are admissible with prior $\frac{1}{2}$ if and only if there exist λ_i with the mean $\frac{1}{2}$ such that $\frac{\mathrm{d}\lambda_i^\ell}{\mathrm{d}\lambda_i}(x) = 2x$ (i.e., $\lambda_i^\ell = \gamma_{\lambda_i}$) and $\frac{\mathrm{d}\lambda_i^h}{\mathrm{d}\lambda_i}(x) = 2(1-x)$. By (30), we can restrict maximization to $\lambda_i^\ell([0,x]) = \lambda_i^h([1-x,1])$, which corresponds to λ_i symmetric around $\frac{1}{2}$, and we obtain (31). $\square$

Proof of Lemma 3. Since B is agent-symmetric, we can use formula (29) from Lemma (6) to compute the value of B, i.e., the transportation problems have equal marginals $\lambda_1^\omega = \lambda_2^\omega$. For supermodular utilities, the optimal transportation plan is given by the co-monotone coupling (10); since the marginals are identical, such coupling is given by a distribution on the diagonal of $[0,1]^2$. We obtain that

$$\mathrm{Val}[B] = \sup_{\substack{\textit{admissible } (\lambda_1^\ell, \lambda_1^h, \lambda_2^\ell, \lambda_2^h) \\ \textit{such that } \lambda_1^\omega = \lambda_2^\omega}} p \cdot \int_{[0,1]} v^\ell(x,x)\, \mathrm{d}\lambda^\ell(x) + (1-p) \cdot \int_{[0,1]} v^\ell(x,x)\, \mathrm{d}\lambda^\ell(x).$$

Comparing this formula to the formula for the value of a single-receiver persuasion problem, we see that the persuasion problem B is equivalent to the single-receiver problem problem with the same prior and the utility $v'^\omega(x) = v^\omega(x,x)$. For single-receiver problems with state-independent utility, the value coincides with the concavification of this utility Kamenica and Gentzkow (2011). We conclude that $\mathrm{Val}[B] = \mathrm{cav}[\overline{v'}](p)$. Since in a single-receiver problem, $|\Omega|$ signals are enough for optimal persuasion, in B, it is enough to consider information structures that reveal the same information to both receivers (i.e., send public signals) and use two signals only. $\square$

B Proofs for Section 4

B.1 Proof of Theorem 2

Dual representation for the value. To prove the dual representation for the value (11) of the persuasion problem B, we introduce an auxiliary zero-sum game G such that the max inf-value of G coincides with the value of B and then exchange max and inf by Sion's minimax theorem.

By Theorem 1, to get the value of B, it is enough to maximize

$$\sum_{\omega\in\Omega} p(\omega)\cdot\int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)} v^\omega(x_1,\ldots x_n)\mathrm{d}\mu^\omega(x_1,\ldots,x_n) \tag{33}$$

over a family of measures $(\mu^\omega)_{\omega\in\Omega}\subset\Delta\big(\Delta(\Omega)\times\ldots\times\Delta(\Omega)\big)$ with admissible marginals. The admissibility constraints require that that the Radon-Nikodym derivatives of the marginals μ_i^ω of μ^ω with respect to some $\lambda_i\in\Delta(\Delta(\Omega))$ satisfy $\frac{\mathrm{d}\mu_i^\omega}{\mathrm{d}\lambda_i}(x_i)=\frac{x_i(\omega)}{p(\omega)}$. From this equation we conclude that $\lambda_i=\sum_{\omega'\in\Omega}p(\omega')\cdot\mu_i^{\omega'}$ and, hence, the admissibility is equivalent to the identity

$$p(\omega)\,\mathrm{d}\mu_i^\omega(x_i)-x_i(\omega)\cdot\sum_{\omega'\in\Omega}p(\omega')\,\mathrm{d}\mu_i^{\omega'}(x_i)=0,$$

which can be rewritten in the integrated form as follows:

$$p(\omega)\cdot\int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)}\psi_i^\omega(x_i)\mathrm{d}\mu^\omega(x_1,\ldots,x_n)-\int_{\Delta(\Omega)}x_i(\omega)\cdot\psi_i^\omega(x_i)\left(\sum_{\omega'\in\Omega}p(\omega')\mathrm{d}\mu^{\omega'}(x_1,\ldots,x_n)\right)=0 \tag{34}$$

for all continuous functions $\psi_i^\omega:\Delta(\Omega)\to\mathbb{R}$.

Let us define the game G. In this game, the maximizer aims to maximize (33) and we allow her to pick an arbitrary collection of probability measures $(\mu^\omega)_{\omega\in\Omega}$, which may have non-admissible marginals. However, the minimizer can penalize her for violation of the identity (34) by selecting a family of continuous functions $(\psi_i^\omega)_{i\in N,\omega\in\Omega}$. The payoff function is defined as follows

$$\begin{aligned}G\Big[\big(\mu^\omega\big)_{\omega\in\Omega},\big(\psi_i^\omega\big)_{i\in N,\omega\in\Omega}\Big]=\sum_{\omega\in\Omega}\Bigg(&p(\omega)\cdot\int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)}v^\omega(x_1,\ldots x_n)\mathrm{d}\mu^\omega(x_1,\ldots,x_n)-\\&-\sum_{i\in N}\Bigg(p(\omega)\cdot\int_{\Delta(\Omega)\times\ldots\times\Delta(\Omega)}\psi_i^\omega(x_i)\,\mathrm{d}\mu^\omega(x_1,\ldots,x_n)-\\&-\int_{\Delta(\Omega)}x_i(\omega)\cdot\psi_i^\omega(x_i)\left(\sum_{\omega'\in\Omega}p(\omega')\mathrm{d}\mu^{\omega'}(x_1,\ldots,x_n)\right)\Bigg)\Bigg)\end{aligned}$$

If the maximizer selects $(\mu^\omega)_{\omega\in\Omega}$ with admissible marginals, then the last two integrals are zero. On the other hand, if the admissibility constraint is violated, the minimizer can make the payoff arbitrary low by picking $(\psi_i^\omega)_{i\in N,\omega\in\Omega}$. Therefore,

$$\mathrm{Val}[B] = \sup_{(\mu^\omega)_{\omega\in\Omega}} \inf_{(\psi_i^\omega)_{i\in N,\omega\in\Omega}} G\Big[\big(\mu^\omega\big)_{\omega\in\Omega}, \big(\psi_i^\omega\big)_{i\in N,\omega\in\Omega}\Big].$$

The assumptions of Sion's minimax theorem[9] are satisfied by $G\Big[\big(\mu^\omega\big)_{\omega\in\Omega}, \big(\psi_i^\omega\big)_{i\in N,\omega\in\Omega}\Big]$ and we can exchange $\sup_{(\mu^\omega)_{\omega\in\Omega}}$ and $\inf_{(\psi_i^\omega)_{i\in N,\omega\in\Omega}}$. Indeed, the set of probability measures on a compact metric space is compact in the weak topology, G is an affine function of strategies of each of the players (and thus both convex and concave), it is an upper semicontinuous function of $(\mu^\omega)_{\omega\in\Omega}$ in the weak topology (see Lemma 4.3 in Section 4 of Villani (2008)) and a continuous function of $(\psi_i^\omega)_{i\in N,\omega\in\Omega}$ in the topology induced by the sup-norm on continuous functions. We obtain

$$\mathrm{Val}[B] = \inf_{(\psi_i^\omega)_{i\in N,\omega\in\Omega}} \sup_{(\mu^\omega)_{\omega\in\Omega}} G\Big[\big(\mu^\omega\big)_{\omega\in\Omega}, \big(\psi_i^\omega\big)_{i\in N,\omega\in\Omega}\Big].$$

For a compact metric space X, we have $\max_{\nu\in\Delta(X)}\int h(x)d\nu(x) = \max_{x\in X} h(x)$ for any upper semicontinuous function h on X; in particular both maxima are attained. Hence the internal unconstrained maximization over $(\mu^\omega)_{\omega\in\Omega}$ leads to the pointwise maxima of the corresponding integrands and we get

$$\mathrm{Val}[B] = \inf_{(\psi_i^\omega)_{i\in N,\omega\in\Omega}} \sum_{\omega\in\Omega} p(\omega)\cdot \max_{(x_i)_{i\in N}\subset\Delta(\Omega)} \left(v^\omega(x_1,\ldots,x_n) - \sum_{i\in N}\left(\psi_i^\omega(x_i) - \sum_{\omega'\in\Omega} x_i(\omega')\cdot\psi_i^{\omega'}(x_i)\right)\right).$$

For a family functions $(\psi_i^\omega)_{i\in N,\omega\in\Omega}$ define a new family $(\varphi_i^\omega)_{i\in N,\omega\in\Omega}$ by the formula

$$\varphi_i^\omega(x_i) = \psi_i^\omega(x_i) - \sum_{\omega'\in\Omega} x(\omega')\cdot\psi_i^{\omega'}(x_i), \qquad x\in\Delta(\Omega). \tag{35}$$

The new family satisfies an extra condition

$$\sum_{\omega\in\Omega} x_i(\omega)\varphi_i^\omega(x_i) = 0, \qquad x\in\Delta(\Omega), \tag{36}$$

and gives the same value to the objective as the original one. We obtain

$$\mathrm{Val}[B] = \inf_{\substack{\text{continuous}\\ (\varphi_i^\omega)_{i\in N,\omega\in\Omega}\text{ such that}\\ \sum_{\omega\in\Omega} x_i(\omega)\varphi_i^\omega(x_i)\equiv 0}} \sum_{\omega\in\Omega} p(\omega)\cdot \max_{(x_i)_{i\in N}\subset\Delta(\Omega)} \left(v^\omega(x_1,\ldots,x_n) - \sum_{i\in N}\varphi_i^\omega(x_i)\right). \tag{37}$$

[9]Sion's theorem claims that $\sup_{x\in X}\inf_{y\in Y} G(x,y) = \inf_{y\in Y}\sup_{x\in X} G(x,y)$ if X and Y are convex subsets of linear topological spaces, at least one of them is compact, and G is an upper semicontinuous quasiconcave function of the first argument and lower semicontinuous quasiconvex of the second. See Mertens et al. (2015), Chapter I.1.

Finally, we we pick arbitrary $V^\omega \geqslant \max_{(x_i)_{i\in N}\subset\Delta(\Omega)} \left(v^\omega(x_1,\ldots,x_n) - \sum_{i\in N}\varphi_i^\omega(x_i)\right)$ and obtain

$$\mathrm{Val}[B] = \inf_{\substack{V^\omega\in\mathbb{R},\ \text{continuous } \varphi_i^\omega \text{ on } \Delta(\Omega) \text{ such that} \\ v^\omega(x_1,\ldots,x_n)\leqslant V^\omega+\sum_{i\in N}\varphi_i^\omega(x_i) \\ \text{and } \sum_{\omega\in\Omega} x_i(\omega)\varphi_i^\omega(x_i)=0}} \sum_{\omega\in\Omega} p(\omega)V^\omega. \tag{38}$$

which coincides with the desired formula from the statement of Theorem 2.

Existence of optima. Here we demonstrate that for continuous utility functions v^ω the infimum in (38) is attained, i.e., optimal $(V^\omega,\ \varphi_i^\omega)_{i\in N,\omega\in\Omega}$ exist.

The idea is to show that we can restrict the minimization to some compact set and then to extract a subsequence converging to an optimum. The restrictions that we can impose on $(V^\omega)_{\omega\in\Omega}$ and φ_i^ω are presented in the following lemma. To formulate it, we define the norm of the utility function by

$$\|v\|_\infty = \max_{\omega\in\Omega,\ x_1,\ldots,x_n\in\Delta(\Omega)} \left|v^\omega(x_1,\ldots,x_n)\right|$$

and its modulus of continuity, by

$$D_v(\varepsilon) = \max_{\substack{\omega\in\Omega,\, i\in N \\ x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\in\Delta(\Omega) \\ x,x'\in\Delta(\Omega)\,:\,|x-x'|\leqslant\varepsilon}} \left|v^\omega\big(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n\big) - v^\omega\big(x_1,\ldots,x_{i-1},x',x_{i+1},\ldots,x_n\big)\right|,$$

where $|x-x'| = \sum_{\omega\in\Omega}|x(\omega)-x(\omega')|$ is the total-variation distance between x and $x'\in\Delta(\Omega)$.

Lemma 7. *Restricting the minimization in (38) to $(V^\omega,\ \varphi_i^\omega)_{i\in N,\omega\in\Omega}$ such that*

$$-\|v\|_\infty \leqslant \left|V^\omega\right| \leqslant \frac{2-p(\omega)}{p(\omega)}\cdot\|v\|_\infty, \tag{39}$$

$$-\frac{2n}{p(\omega)}\cdot\|v\|_\infty \leqslant \left|\varphi_i^\omega(x)\right| \leqslant \frac{2}{p(\omega)}\cdot\|v\|_\infty, \qquad x\in\Delta(\Omega), \tag{40}$$

$$\left|\varphi_i^\omega(x)-\varphi_i^\omega(x')\right| \leqslant 2\cdot D_v\Big(|x-x'|\Big) + \frac{2n}{\min_{\omega'\in\Omega} p(\omega')}\cdot\|v\|_\infty\cdot|x-x'|, \qquad x,\,x'\in\Delta(\Omega), \tag{41}$$

does not affect the optimal value.

We first check that this lemma implies the existence of the optimal $(V^\omega,\ \varphi_i^\omega)_{i\in N,\omega\in\Omega}$ and then prove the lemma. Consider a sequence $(V^{\omega,t},\ \varphi_i^{\omega,t})_{i\in N,\omega\in\Omega}$ indexed by a parameter $t=1,2,\ldots$ and such that the objective in (38) converges to its optimum along this sequence, as t goes to infinity. By Lemma 7, we can additionally require that $(V^{\omega,t},\ \varphi_i^{\omega,t})_{i\in N,\omega\in\Omega}$ satisfy the conditions (39), (40) and (41) for each t. The set of numbers defined by (39) is compact as a closed bounded subset of $\mathbb{R}^\Omega$. Functions satisfying (40) and (41) are uniformly bounded and uniformly continuous and thus, by the Arzelà–Ascoli theorem (see Rudin (1964)), this class of functions compact in the topology of the space of continuous functions (induced by the sup-norm). The product of compact sets is compact and, hence, the sequence $(V^{\omega,t},\, \varphi_i^{\omega,t})_{i\in N,\omega\in\Omega}$ belongs to a compact set. Let us extract a converging subsequence and denote its limit by $(V^{\omega},\, \varphi_i^{\omega})_{i\in N,\omega\in\Omega}$. The objective in (38) is continuous and the constraints are closed. Hence the collection $(V^{\omega},\, \varphi_i^{\omega})_{i\in N,\omega\in\Omega}$ gives the optimal value to the objective, satisfies the constraints, and thus is optimal.

To complete the proof of Theorem 2 it remains to prove the lemma.

Proof of Lemma 7. For a given family $(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}$ of continuous functions satisfying (36), let $V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big]$ be the minimal value of V^{ω} such that $(V^{\omega},\, \varphi_i^{\omega})$ satisfy the constraints of (38):

$$V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big] = \max_{(x_i)_{i\in N}\subset\Delta(\Omega)} \left(v^{\omega}(x_1,\ldots,x_n) - \sum_{i\in N} \varphi_i^{\omega}(x_i) \right). \tag{42}$$

Without loss of generality, we can assume that V^{ω} in (38) is given by $V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big]$ and, hence, V^{ω} is determined by functions $(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}$, which remain the only free parameter in the minimization. In particular, to prove the bounds (39) on V^{ω} it is enough to show that we can restrict minimization to $(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}$ such that

$$-\|v\|_{\infty} \leqslant \left|V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big]\right| \leqslant \frac{2-p(\omega)}{p(\omega)} \cdot \|v\|_{\infty}. \tag{43}$$

Recall that $\delta_\omega \in \Delta(\Omega)$ is the point mass at the state ω. Plugging $x_i = \delta_\omega$ for each i in (42), we obtain the following lower bound:

$$V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big] \geqslant v^{\omega}\big(\delta_\omega,\ldots,\delta_\omega\big) \geqslant -\|v\|_{\infty}.$$

Hence, the lower bound in (43) holds.

The optimal value of (38) cannot exceed the best value of the objective attained at the zero functions φ_i^{ω}. Hence, the the minimization can be restricted to $(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}$ such that

$$\sum_{\omega\in\Omega} p(\omega)\cdot V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big] \leqslant \sum_{\omega\in\Omega} p(\omega)\cdot V^{\omega}\big[(0)_{i\in N,\omega\in\Omega}\big].$$

Since the right-hand side does not exceed $\|v\|_{\infty}$, we get

$$\sum_{\omega\in\Omega} p(\omega)\cdot V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big] \leqslant \|v\|_{\infty}. \tag{44}$$

Changing all summands on the left-hand side of (44) except one to their lower bounds and transferring them to the right-hand side, we get

$$V^{\omega}\big[(\varphi_i^{\omega})_{i\in N,\omega\in\Omega}\big] \leqslant \frac{2-p(\omega)}{p(\omega)} \cdot \|v\|_{\infty}. \tag{45}$$

We obtain the upper bound in (43). Moreover, this inequality implies an upper bound on φ_i^ω. Indeed, let us plug $x_j = \delta_\omega$ for all receivers j except $j = i$ in the objective of (42). The value of the objective on this input cannot exceed the optimal value and, taking into account that $\varphi_j^\omega(\delta_\omega) = 0$ thanks to (36), we deduce

$$v^\omega(\delta_\omega, \ldots, \delta_\omega, x_i, \delta_\omega, \ldots, \delta_\omega) + \varphi_i^\omega(x_i) \leqslant V^\omega\big[(\varphi_i^\omega)_{i\in N, \omega\in\Omega}\big].$$

Consequently,

$$\varphi_i^\omega(x) \leqslant \frac{2}{p(\omega)} \cdot \|v\|_\infty, \tag{46}$$

i.e, the upper bound in (40) holds.

Let us summarize: without loss of generality, the minimization in (38) can be restricted to families of continuous functions $(\varphi_i^\omega)_{i\in N,\omega\in\Omega}$ satisfying (36) and (44); the upper bound (46) is satisfied for all such families automatically as well as the bounds (43). Now, we consider such a family, fix a receiver $k \in N$, and show that we can replace the functions $(\varphi_k^\omega)_{\omega\in\Omega}$ by $(\tilde{\tilde{\varphi}}_k^\omega)_{\omega\in\Omega}$ keeping the rest of the family unchanged in a way that the new family satisfies the same requirements, the value of the objective remains the same or improves, and, most importantly, the functions $(\tilde{\tilde{\varphi}}_k^\omega)_{\omega\in\Omega}$ additionally satisfy bounds (40) and (41). Define $\tilde{\varphi}_k^\omega$ by

$$\tilde{\varphi}_k^\omega(x) = \max_{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n} \left(v^\omega(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n) - \sum_{i\in N\setminus\{k\}} \varphi_i^\omega(x_i) \right) - V^\omega\big[(\varphi_i^\omega)_{i\in N,\omega\in\Omega}\big].$$

From the definition we see that

$$V^\omega\Big[\Big((\tilde{\varphi}_k^\omega)_{\omega\in\Omega}, (\varphi_i^\omega)_{i\in N\setminus\{k\},\omega\in\Omega}\Big)\Big] = V^\omega\big[(\varphi_i^\omega)_{i\in N,\omega\in\Omega}\big]$$

and, moreover, the functions $\tilde{\varphi}_k^\omega$ are pointwise minimal among all the functions with this property. Hence, $\varphi_k^\omega \geqslant \tilde{\varphi}_k^\omega$.

The functions $(\tilde{\varphi}_k^\omega)_{\omega\in\Omega}$ may violate the requirement (36). To enforce this requirement we set

$$\tilde{\tilde{\varphi}}_k^\omega(x) = \tilde{\varphi}_k^\omega(x) - \sum_{\omega'\in\Omega} x(\omega') \cdot \tilde{\varphi}_k^{\omega'}(x).$$

The functions $\tilde{\tilde{\varphi}}_k^\omega$ satisfy (36). Since $\varphi_k^\omega \geqslant \tilde{\varphi}_k^\omega$,

$$\sum_{\omega'\in\Omega} x(\omega') \cdot \tilde{\varphi}_k^\omega(x) \leqslant \sum_{\omega'\in\Omega} x(\omega') \cdot \varphi_k^\omega(x) = 0,$$

and we see that $\tilde{\tilde{\varphi}}_k^\omega \geqslant \tilde{\varphi}_k^\omega$. Therefore,

$$V^\omega\Big[\Big((\tilde{\tilde{\varphi}}_k^\omega)_{\omega\in\Omega}, (\varphi_i^\omega)_{i\in N\setminus\{k\},\omega\in\Omega}\Big)\Big] \leqslant V^\omega\Big[\Big((\tilde{\varphi}_k^\omega)_{\omega\in\Omega}, (\varphi_i^\omega)_{i\in N\setminus\{k\},\omega\in\Omega}\Big)\Big],$$

and so replacing φ_k^ω by $\widetilde{\widetilde{\varphi}}_k^\omega$ can only improve the objective in (37).

We conclude that the constructed family satisfies the conditions (36) and (44) (hence, the upper bound (46) also holds) and the value of the objective remains the same or improves. Now let us check that $\widetilde{\widetilde{\varphi}}_k^\omega$ satisfies the lower bound in (40) and the bound (41).

From the definition of $\widetilde{\varphi}_k^\omega$ the bounds (45) and (46), we obtain

$$-\frac{2n}{p(\omega)} \cdot \|v\|_\infty \leqslant \widetilde{\varphi}_k^\omega(x).$$

Since $\widetilde{\widetilde{\varphi}}_k^\omega \geqslant \widetilde{\varphi}_k^\omega$, the same lower bound holds for $\widetilde{\widetilde{\varphi}}_k^\omega$. Thus $\widetilde{\widetilde{\varphi}}_k^\omega$ satisfies both bounds of (40).

To prove (41), we estimate the difference $\left|\widetilde{\varphi}_k^\omega(x) - \widetilde{\varphi}_k^\omega(x')\right|$ first. By the definition of $D_v(\varepsilon)$,

$$v^\omega\big(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n\big) + D_v\Big(|x - x'|\Big) \geqslant v^\omega\big(x_1, \ldots, x_{i-1}, x', x_{i+1}, \ldots, x_n\big)$$

for any $x, x' \in \Delta(\Omega)$ and all $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in \Delta(\Omega)$. Subtracting $\sum_{i \in N\setminus\{k\}} \varphi_i^\omega(x_i) + V^\omega\big[(\varphi_i^\omega)_{i\in N, \omega\in\Omega}\big]$ from both sides and taking maximum over $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in \Delta(\Omega)$, we get

$$\widetilde{\varphi}_k^\omega(x) + D_v\Big(|x - x'|\Big) \geqslant \widetilde{\varphi}_k^\omega(x').$$

Combining this inequality with the one where the roles of x and x' are exchanged, we obtain

$$\left|\widetilde{\varphi}_k^\omega(x) - \widetilde{\varphi}_k^\omega(x')\right| \leqslant D_v\Big(|x - x'|\Big). \tag{47}$$

From the definition of $\widetilde{\widetilde{\varphi}}_k^\omega$,

$$\widetilde{\widetilde{\varphi}}_k^\omega(x) - \widetilde{\widetilde{\varphi}}_k^\omega(x') = \Big(\widetilde{\varphi}_k^\omega(x) - \widetilde{\varphi}_k^\omega(x')\Big) - \sum_{\omega'\in\Omega} x(\omega')\Big(\widetilde{\varphi}_k^{\omega'}(x) - \widetilde{\varphi}_k^{\omega'}(x')\Big) - \sum_{\omega'\in\Omega} \Big(x(\omega') - x'(\omega')\Big)\widetilde{\varphi}_k^{\omega'}(x').$$

Estimating the first two terms on the right-hand side using (47) and bounding the absolute value of the last term by $|x - x'| \cdot \max_{\omega', x} |\widetilde{\varphi}_k^{\omega'}(x)|$, we see that $\widetilde{\widetilde{\varphi}}_k^\omega$ satisfies (41).

Sequentially replacing φ_k^ω in $(\varphi_i^\omega)_{i\in N,\omega\in\Omega}$ by $\widetilde{\widetilde{\varphi}}_k^\omega$ for all receivers $k \in N$, we obtain a collection of functions that satisfies (40) and (41), while the value of the objective in (38) remains the same or improves. Thus restricting the minimization in (38) to families satisfying (40) and (41) does not affect the optimal value. □

B.2 Proof of Theorem 3

Consider a persuasion problem $B = (\Omega, p, N, v)$ and a new problem $B' = (\Omega, p, N, u)$ such that $u \geqslant v$ and revealing no information is optimal in (Ω, q, N, u) for any q. Our goal is to show that $\mathrm{Val}[B] = \inf_u \mathrm{Val}[B']$. Since, $u \geqslant v$, the value of B' is at least as high as the value of B. Hence, it remains to demonstrate that for any $\varepsilon > 0$, we can find u such that $\mathrm{Val}[B'] \leqslant \mathrm{Val}[B] + \varepsilon$.

By Theorem 2, we can find $V_B^\omega \in \mathbb{R}$ and continuous functions $\varphi_{B,i}^\omega$ on $\Delta(\Omega)$ such that

$$\mathrm{Val}[B] \leqslant \sum_{\omega\in\Omega} p(\omega) V_B^\omega \leqslant \mathrm{Val}[B] + \varepsilon, \tag{48}$$

$v^\omega(x_1, \ldots, x_n) \leqslant V_B^\omega + \sum_{i\in N} \varphi_{B,i}^\omega(x_i)$, and $\sum_{\omega\in\Omega} x_i(\omega)\varphi_{B,i}^\omega(x_i) = 0$. Define u as follows:

$$u^\omega(x_1, \ldots, x_n) = V_B^\omega + \sum_{i\in N} \varphi_{B,i}^\omega(x_i).$$

By the construction $u \geqslant v$. Applying Theorem 2 to the persuasion problem (Ω, q, N, u), we see that its value cannot exceed $\sum_{\omega\in\Omega} q(\omega) V_B^\omega$, the value of the objective achieved if we pick $V^\omega = V_B^\omega$ and $\varphi_i^\omega = \varphi_{B,i}^\omega$. However, if the sender reveals no information, the sender's expected utility is equal to $\sum_{\omega\in\Omega} q(\omega) u^\omega(q, \ldots, q) = \sum_{\omega\in\Omega} q(\omega) V_B^\omega$. We conclude that

$$\mathrm{Val}\Big[(\Omega, q, N, u)\Big] = \sum_{\omega\in\Omega} q(\omega) V_B^\omega$$

and revealing no information is optimal for the sender. Thus the persuasion problem $B' = (\Omega, p, N, u)$ satisfies all the requirements and, by (48), the value of B' is bounded by $\mathrm{Val}[B] + \varepsilon$. We conclude that $\mathrm{Val}[B] = \inf_u \mathrm{Val}[B']$.

Note that for continuous v, the infimum is achieved because for such v it is achieved in Theorem 2 and we can take $\varepsilon = 0$ in the above construction.