Mechanism Design: lecture 4
Fair Division

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Motivation

Auctions = simple and efficient way to distribute resources. Why do we need something else?

Monetary transactions may be ruled out by

- ethical reasons. Bias towards richest ⇒ repugnant
  Examples: government programs (education, social housing), charity, organ transplants
- institutional reasons. Who is auctioneer?
  Examples: division of a common property (partners dissolving their partnership, inheritance), allocation of tasks or resources within the firm (office space, IT facilities, bonuses), division of a common surplus among business-partners

How to distribute resources if we can't auction them? What is fairness and how to take it into account?

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Fair division problems and their types

Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences. Aristotle, Ethics.

Agents may differ in their

- rights
  Example: one partner contributed to the project more than others and hence deserves bigger share of surplus
- tastes (preferences)
  Example: Alice wants to attend an Archeology class and does not want to learn Economics, but Bob does.

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The two extreme cases:

- Unequal rights & identical tastes
  Cooperative game theory is about that $\Rightarrow$ we will not discuss
- Equal rights & different tastes

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- **Equal rights & different tastes**

We focus on the last extreme case and aim to understand how individual tastes can be taken into account.

FD history (equal rights & different tastes)

- 1950s: first paper on **cake-cutting** by Hugo Steinhaus
  cake = divisible inhomogeneous resource (land or time)
  We will not discuss. Why?
  - Most of the results are focused on fairness without efficiency $\Rightarrow$ criticized by economists
  - Not much realistic

  But there are many interesting results in the last decade.

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    Examples: inheritance, common property between partners, seats in overdemanded courses, etc
  - Cons: Wait... Usually the goods are indivisible!

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Lifehack: what is 0.3 of a bicycle?

- randomization: getting the bicycle with probability 0.3
- time-sharing: using bicycle 30% of time

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- last 50 years: **fair division of divisible private goods** Today!
- last decade: **indivisibilities** on Wednesday!
  - Pros: realistic + no lifehacks
  - Cons: non-trivial normative and algorithmic questions

Outline

- The model
- Efficiency and Fairness
- Social Welfare maximizers: Utilitarian, Egalitarian, and the Nash rules
- Equal choice opportunities and the Competitive Rule

The model

The model. Fair division of divisible goods

- $N = \{1, 2, 3.., n\}$ a set of agents
- $A = \{a, b, c, ..\}$ a set of divisible private goods
- $\omega \in R_+^A$ a social endowment
  - w.l.o.g. $\omega = (1, 1, ..1, 1)$
- $z_i = (z_{ia}, z_{ib}, z_{ic}..) \in R_+^A$ is a bundle of goods received by agent i
- an allocation $z = (z_i)_{i \in N}$ is a collection of bundles z_i of all agents with the condition that all goods are distributed: $\sum_{a \in A} z_i = \omega$
- Preferences on bundles are given by utility functions (cardinal setup):
  - $U_i(z_i)$ is agent i's utility
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Examples?

- give everything to the first agent
- equal division $z_i = \omega / |N|$

The domain of preferences

We will assume that agents have

additive (aka linear) utilities:

$U_i(z_i) = u_{ia}z_{ia} + u_{ib}z_{ib} + u_{ic}z_{ic} + ... =< u_i, z_i >$

- **normalization** $< u_i, \omega >= 1$ (or 1000), i.e., does not depend on i

Pros: easy to report and represent $\Rightarrow$ wide use

$$\begin{matrix} U_1(z_1) = 2z_{1a} + 3z_{1b} + 5z_{1c} \\ U_2(z_2) = 7z_{2a} + 1z_{2b} + 2z_{2c} \end{matrix} \iff \begin{matrix} & a & b & c \\ u_1: & 2 & 3 & 5 \\ u_2: & 7 & 1 & 2 \end{matrix}$$

Cons: rules out complementarities between items

Other relevant domains (that we will not discuss):

- Leontief $U_i(z_i) = \min_a u_{ia}z_{ia}$: items are perfect complements
- CES : compromise between additive and Leontief
- Arrow-Debreu : too general (hard to report)
- future lectures by Alex: rankings instead of utilities (ordinal setup)

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Roles of efficiency and fairness. Example

	books	*flowers*
u_{Alice} :	70	30
u_{Bob} :	10	90

What do you think of such allocations?

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- equal division $\begin{matrix} z_{Alice}: & 0.5 & 0.5 \\ z_{Bob}: & 0.5 & 0.5 \end{matrix}$ Fair but inefficient. Difference in preferences can be exploited to make both better off.

Efficiency

allocation z is efficient if there is no other allocation z' such that $U_i(z'_i) \geq U_i(z_i)$ for all i and for at least one i the inequality is strict.

Inefficiency gives an opportunity to trade: both are happy to exchange less wanted items (a deep idea to be exploited in 2 hours)

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- give an item to an agent that values it most (utilitarian rule)

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Conclusion: We need fairness and efficiency at the same time. But what is fairness?

Fairness

The two dominant criteria in Economics:

Fair Share Guaranteed (aka Equal Division Lower Bound)

Every agent prefers an allocation z to the equal division:

$$U_i(z_i) \geq U_i(\omega)/|N|$$

Envy-Freeness

Every agent prefers his bundle to the bundle of every other agent:

$$U_i(z_i) \geq U_i(z_j)$$

for all $i, j \in N$.

Guess:

- What is stronger, FSG or E-F?
- Do Efficient + E-F allocations always exist?

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- Do Efficient + E-F allocations always exist? We will see soon

Social Welfare maximizers: Utilitarian, Egalitarian, and the Nash rules

SW maximizers

Widespread idea: to get an efficient allocation let's maximize Social Welfare

Lemma

If an allocation z maximizes $g(U_1(z_1), U_2(z_2)..U_n(z_n))$, where g is strictly increasing in each variable $\Rightarrow z$ is efficient.

Denote such a rule by f_g.

Our plan: check f_g for fairness for different g.

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The Utilitarian rule

$$f_{UT} \text{ outputs } z : \quad \sum_{i \in N} U_i(z_i) \to \max$$

- Depends on normalization of u. Don't forget to normalize.
- Wide use in Economics
- Philosophy background: Jeremy Bentham (1748 — 1832): *"The goal of the society is the greatest happiness of the greatest number of its members"*

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	book	*flower*	*bicycle*	*laptop*	*armchair*
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Alice gets only 5% from 100%. FSG says that she should get at least 50%, so f_{UT} violates FSG and E-F.

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- Philosophy background: Jeremy Bentham (1748 — 1832): *"The goal of the society is the greatest happiness of the greatest number of its members"*

Example:

	beer	*wine*	*vodka*
u_{Alice} :	80	10	10
u_{Bob} :	10	80	10
u_{Claire} :	10	10	80
u_{Dave} :	33	33	34

The Utilitarian rule

$$f_{UT} \text{ outputs } z : \quad \sum_{i \in N} U_i(z_i) \to \max$$

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Flexible agents may get nothing!

Conclusion: f_{UT} is a very unfair rule (until there are monetary transfers to compensate unlucky agents)

The Egalitarian rule

$$f_{Egal} \text{ outputs } z: \quad \min_{i \in N} U_i(z_i) \to \max$$

- introduced by Pazner and Schmeidler [1]
- Philosophy background: John Rawls (1921 – 2002) : *"The goal of the society is the greatest happiness of the least happy members"*

Properties (assume that the matrix u has no zeros):

- equitability: $U_i(z_i) = U_j(z_j)$
- Efficiency
- FSG?
- E-F?

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- equitability: $U_i(z_i) = U_j(z_j)$ *Proof:* if not, we can transfer a small amount of some good from the happiest agent to all others thus increasing the Egalitarian SW.
- Efficiency
- FSG?
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- E-F?

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Properties (assume that the matrix u has no zeros):

- equitability: $U_i(z_i) = U_j(z_j)$
- Efficiency
- FSG? Yes, because Egalitarian SW for the equal division allocation is $1/n$.
- E-F?

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- FSG?
- E-F? For $|N| > 2$, no. Example in 5 min.

[1] Pazner, E. A., Schmeidler, D. (1978). Egalitarian equivalent allocations: A new concept of economic equity. The Quarterly Journal of Economics, 92(4), 671-687.

How to compute f_{Egal} for 2 agents? Example

	a	b	c
u_{Alice} :	1	20	9
u_{Bob} :	15	5	10

Useful tool: the structure of efficient allocations for 2 agents

Rearrange items s.t. u_{1a}/u_{2a} is decreasing. Then any efficient allocation has the following form

u_{Alice} :	1	1	...	1	x	0	0	...	0
u_{Bob} :	0	0	...	0	$1-x$	1	1	...	1

Proof: other allocations provide opportunities to trade.

Corollary: any efficient rule for two agents splits at most one item, i.e., produces almost-indivisible allocations! For n agents there are at most $n-1$ splits.

To compute f_{Egal} we need to find an efficient allocation such that $U_1 = U_2$.

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To compute f_{Egal} we need to find an efficient allocation such that $U_1 = U_2$.

f_{Egal} violates E-F. Example with 3 agents

$$\begin{array}{rcc} & a & b \\ u_{Alice}: & 6 & 6 \\ u_{Bob}: & 8 & 4 \\ u_{Claire}: & 9 & 3 \end{array}$$

By similar "trading argument" see that any Efficient allocation has the form

$$\begin{array}{rcc} & a & b \\ z_{Alice}: & 0 & x \\ z_{Bob}: & 1-y & 1-x \\ z_{Claire}: & y & 0 \end{array}$$

it remains to find x and y from

$$6x = 8(1-y) + 4(1-x) = 9y$$

to see that Claire envies Bob.

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The Nash rule aka NashMaxProduct or NMP rule

A compromise between Utilitarian and Egalitarian approaches:

$$f_{Nash} \text{ outputs } z: \quad \prod_{i \in N} U_i(z_i) \to \max$$

a similar rule was introduced by J. Nash in the context of axiomatic bargaining[2]

Properties:

- Efficiency
- FSG? Yes!
- Envy-Freeness? Yes! *Proof:* see the blackboard

[2]Nash, John (1950). The Bargaining Problem. Econometrica. 18(2): 155–162. JSTOR 1907266

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There are many confirmations that f_{Nash} is **the best rule to divide goods under additive utilities**. But why is the Nash product so specific? Wait 10min: there is a deep explanation based on theory of General Equilibrium.

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How to compute f_{Nash} for 2 agents?

Example:

$$\begin{array}{rcc} & a & b \\ u_{Alice}: & 9 & 1 \\ u_{Bob}: & 6 & 4 \end{array}$$

Let's use the structure of efficient allocations again:

- for every allocation

$$\begin{array}{rcccccccccc} u_{Alice}: & 1 & 1 & \dots & 1 & x & 0 & 0 & \dots & 0 \\ u_{Bob}: & 0 & 0 & \dots & 0 & 1-x & 1 & 1 & \dots & 1 \end{array}$$

  find x such that first order conditions for the maximum of the product are satisfied; if such x belongs to $[0, 1]$ we are done!
- if no such x found, the Nash product is maximized by one of "indivisible" allocations (those with $x = 0$). Check them all and find the optimal.

Equal choice opportunities and the Competitive Rule

Envy-freeness as equal choice opportunities

Alice spends 100$ in a mall, so does Bob. Will they envy each other?

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No, because both select the best bundle of goods from the same choice set (their budget set).

Microeconomists combined this observation with theory of General Equilibrium $\Rightarrow$ the Competitive Rule[3]

[3] Varian, H. R. (1974). Equity, envy, and efficiency. Journal of economic theory, 9(1), 63-91.

The Competitive Rule f_{CR}

aka Competitive Equilibrium with Equal Incomes (CEEI) or Pseudo-Market mechanism

Informal definition:

- give every agent a unit amount of "virtual" money
- select prices s.t. the market clears, when everybody buys the best bundle he/she can afford

The resulting allocation is

- envy-free $\Longleftarrow$ equal choice opportunities
- efficient $\Longleftarrow$ "invisible hand" of Adam Smith (1st fundamental theorem of Welfare Economics)

And this holds in a very general setup (e.g., in Arrow-Debreu preferences).

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The Competitive Rule f_{CR}

Formal definition

z is a competitive allocation if there is a vector of prices $p \in \mathbb{R}_+^A$ such that for any agent $i \in N$

$$z_i \text{ maximizes } U_i \text{ over the budget set } B(p) = \{y \in \mathbb{R}_+^A : \sum_{a \in A} p_a y_a \leq 1\}.$$

Properties: Envy-Free & Efficient and...

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Properties: Envy-Free & Efficient and... it coincides with the Nash rule!

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Theorem (Eisenberg (1961), Gale (1960))

For homogeneous utilities (in particular, for additive)

$$f_{CR} = f_{Nash}.$$

Example

Find the outcome of f_{CR} and the competitive prices for

$$\begin{array}{rcc} & a & b \\ u_{Alice}: & 9 & 1 \\ u_{Bob}: & 6 & 4 \end{array}$$

Compare with the outcome of the Nash rule.

Tools:

- the structure of efficient allocations + knowledge that the outcome of CR satisfies FSG
- if an agent i consumes two goods a and b "bang per buck" are the same: $\frac{u_{ia}}{p_a} = \frac{u_{in}}{p_b}$
- every agent spends his unit of money completely.

Main points for takeaway:

- Concepts of fairness and their interplay with efficiency
- The Utilitarian rule may be very unfair
- "Virtual" market approach provides fair and efficient mechanism to distribute private goods
- For additive utilities it coincides with the Nash rule

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