Mechanism Design
Fair division with money: rent division between roommates

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So different fair division:

- **Last Friday:** divisible goods (randomization or time sharing)
  fairness concepts; examples of division rules
- **Today:** indivisible goods & one divisible good (money)
  The goal: to illustrate MD ideas in application to the real problem of rent-division
- **On October 11:** indivisible goods
  algorithmic & normative issues

Fair division with money. Motivation

Question

3 friends rent a flat with 3 rooms together. The rooms are different one to another. How can they split the total rent (say, 900$)?

Your ideas?

- Let us divide the rent equally!
- Let us divide the rent proportionally to the area of rooms! `http://www.roomiecalc.com/`
- Let us go further and take additional parameters into account `https://www.splitwise.com/calculators/rent`

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  But what if the "best" room is a communicating room (every person entering the flat should pass through this room)?!?

Fair division with money. Motivation

Question

3 friends rent a flat with 3 rooms together. The rooms are different one to another. How can they split the total rent (say, 900$)?

Conclusion: We can make new services capturing more and more parameters of the flat. But

- there always will be something that is not taken into account
- it is unclear why, for example, for having a private bathroom I will pay 30% more and not 20%?

What Mechanism Design tells us?

- Which room is the best and which is the worst for particular roommate reflects his private values (preferences). So, the private values are important, not the physical parameters of the room:
  - Private values and physical parameters may be consistent but intensity of PV is also important
    - Alice prefers to live in a big room but for an artist Bob a big window is much more important than area
  - Private values and physical parameters may be inconsistent
    - a sociopath Claire prefers a small comfortable room to a big one, where she expects Alice and Bob making parties;
- Which rent division is appropriate is to be determined via normative arguments.
- Rent division is reasonable to combine with allocating the rooms among roommates

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Today: two MD approaches to rent division

- Mechanism from `http://www.spliddit.org/`
  Features:
  - Simple preferences (quasilinear domain)
  - Fairness and Efficiency
  - Links with Competitive Rule
- Mechanism from `https://www.nytimes.com/interactive/2014/science/rent-division-calculator.html`
  Features:
  - General preferences
  - Interactive elicitation procedure
  - Focus on fairness

Mechanism from Spliddit

The model

- $N = \{1, 2, 3\}$ a set of roommates
- $A = \{a, b, c\}$ a set of indivisible rooms, $|N| = |A|$
- R the total rent to be divided
- An allocation is a pair $(\sigma : N \to A, p \in \mathbb{R}^A)$, where
  - $\sigma(i)$ is a room allocated to a roommate i
  - p_a is the rent for a room a
  - $p_a + p_b + p_c = R$
- Agent i reports his values for every room $u_i = (u_i a, u_i b, u_i c)$ with the condition $u_{ia} + u_{ib} + u_{ic} = R$
  - Interpretation: u_{ia} is the appropriate payment for a person leaving in a room a from the point of view of roommate i
- Utility of agent i from getting a room a is quasi-linear: $u_{ia} - p_a$

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$$f : (u_i)_{i \in N} \to (\sigma, p)$$

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Examples?

The goals:

Efficiency

An allocation is **Efficient** if there is no other allocation that is weakly preferred by all agents and by at least one strictly.

Fairness

An allocation (σ, p) is **Envy-Free** if for all $i, j \in N$

$$u_{i\sigma(i)} - p_{\sigma(i)} \geq u_{i\sigma(j)} - p_{\sigma(j)}.$$

Example

Two roommates, two rooms, the rent is 1000$

	bigroom	*smallroom*
u_{Alice} :	700	300
u_{Bob} :	600	400

- Find all efficient allocations
- Find an Envy-Free allocation. Do they exist? Are they efficient?

A selection problem: Which E-F allocation to choose if there are many? A mechanism from Spliddit[1] picks the most Egalitarian among Envy-Free Efficient allocations. Find it for our example.

[1] Gal, Y. A. K., Mash, M., Procaccia, A. D., Zick, Y. (2016, July). Which Is the Fairest (Rent Division) of Them All?. In Proceedings of the 2016 ACM Conference on Economics and Computation (pp. 67-84). ACM.

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Properties of Efficient and Envy-Free allocations

Let's formalize our observations:

Structure of Efficient allocations

An allocation (σ, p) is efficient $\Leftrightarrow$ it is Utilitarian:

$$\sigma \text{ maximizes } SW(\sigma) = \sum_{i \in N} u_{i\sigma(i)}$$

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Proof: let's draw!

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Proof: consider an arbitrary allocation (σ', p'). By Envy-Freeness of (σ, p):

$$u_{i\sigma(i)} - p_{\sigma(i)} \geq u_{i\sigma'(i)} - p_{\sigma'(i)}$$

by summation over i we get

$$SW(\sigma) - R \geq SW(\sigma') - R.$$

□

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Sharp contrast:

- FD without money $\Rightarrow$ the Utilitarian rule is orthogonal to fairness
- FD with monetary compensations $\Rightarrow$ every Envy-Free allocation maximizes the Utilitarian SW

Existence of Envy-Free allocations:

A rent division problem $(N, A, (u_i)_{i \in A})$ defines an Assignment Economy (aka Shapley-Scarf housing markets):

- non-monetary indivisible goods A are traded by agents; private ownership; every agent owns and needs one good $\Rightarrow$ exchange; monetary transfers are allowed

Remark: we do not discuss ownership structure since it does not affect equilibrium allocations

Definition:

(σ, p) is a Walrasian (aka Competitive) Equilibrium in an assignment economy if for any agent i a good $a = \sigma(i)$ received by him maximizes

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Theorem (Aragones, 1992[2])

WE exists for any $(N, A, (u_i)_{i \in A})$

[2]Moulin, H. (2014). Cooperative microeconomics: a game-theoretic introduction. Princeton University Press; page 213

Proof of the Aragones theorem:

- Fix σ : $SW(\sigma) = \sum_{i \in N} u_{i\sigma(i)} \to \max$
- Consider a complete oriented graph Γ with the set of vertices N
- To an edge $e = (i, j)$ assign a weight $w_{i,j} = u_{i\sigma(i)} - u_{i\sigma(j)}$
- The weight $w(l)$ of a path $l = ((i, j), (j, k), (k, l)...)$ is the sum of weights
- Define $p_{\sigma(i)} = \min_{\text{path} l \text{ starting from} i} w(l)$
  - The minimal path visits each vertex at most once since Γ has no negative cycles (otherwise we can improve $SW(\sigma)$)
- Let us check that p defines an equilibrium price. For any path l' starting from j consider a path $l = ((i, j), l')$. We get:

$$p_{\sigma(i)} \leq w_{i,j} + p_{\sigma(j)}$$

or

$$u_{i\sigma(i)} - p_{\sigma(i)} \geq u_{i\sigma j} - p_{\sigma(j)}$$

for any i and j. □

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- To an edge $e = (i, j)$ assign a weight $w_{i,j} = u_{i\sigma(i)} - u_{i\sigma(j)}$
- The weight $w(l)$ of a path $l = ((i, j), (j, k), (k, l) \ldots)$ is the sum of weights
- Define $p_{\sigma(i)} = \min_{\text{path} l \text{ starting from} i} w(l)$
    - The minimal path visits each vertex at most once since Γ has no negative cycles (otherwise we can improve $SW(\sigma)$)
- Let us check that p defines an equilibrium price. For any path l' starting from j consider a path $l = ((i, j), l')$. We get:

$$p_{\sigma(i)} \leq w_{i,j} + p_{\sigma(j)}$$

or

$$u_{i\sigma(i)} - p_{\sigma(i)} \geq u_{i\sigma j} - p_{\sigma(j)}$$

for any i and j. □

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How to use Aragones construction for computing Envy-Free rent division? Example

	a	b	c
u_{Alice} :	400	400	200
u_{Bob} :	500	400	100
u_{Claire} :	700	200	100

Idea:

- Find the utility-maximizing σ
- Draw the graph and compute weights $w_{i,j} = u_{i\sigma(i)} - u_{i\sigma(j)}$
- Find $p = (p_a, p_b, p_c)$
- Normalization. Define $p' = (p_a + \delta, p_b + \delta, p_c + \delta)$, where δ is such that $p'_a + p'_b + p'_c = 1000\$$

Manipulability

On Friday we mentioned that for fair division with cardinal preferences (i.e., utilities) Strategy-Proofness is incompatible with Fairness and Efficiency.

- Private goods without money: a non-trivial result (Zhou 1991, Thomson & Cho 2017)
- Rent division: very easy one...

Proposition

Every Envy-Free rent-division rule is manipulable

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Proof:

$$f\begin{pmatrix} & a & b \\ u_{Alice}: & 100-\alpha & \alpha \\ u_{Bob}: & 100-\beta & \beta \end{pmatrix} = (\sigma, p)$$

- Assume $\alpha < \beta$.
- E-F$\Rightarrow$ Efficiency $\Rightarrow \sigma(A) = a$ and $\sigma(B) = b$.
- E-F implies $100 - \alpha - p_a \geq \alpha - p_b$ and $\beta - p_b \geq 100 - \beta - p_b$.
  Hence
  $$\alpha \leq p_b \leq \beta$$
- Manipulation:
  - If $p_b > \alpha$, then Bob can report β' such that $\alpha < \beta' < p_b$ and pay less than β'
  - If $p_b = \alpha$, then $p_a = 100 - \alpha$ and Alice can report α' such that $\alpha < \alpha' < \beta$ and pay less.

□

Why VCG-mechanisms are not well-suited for rent division?

(generalized)VCG

VCG is defined for agents with quasi-linear utilities. It

- outputs an efficient allocation (σ, t)
- makes being truthful the dominant strategy by clever money-transfers:

$$t_i = SW^*_{-i} - SW_{-i}(\sigma) + h_i(u_{-i}),$$

where SW^* is the optimal SW and h is an arbitrary function
In other words, every agent pays his damage to the social welfare + something independent of his report.

Exercise: Construct a VCG mechanism for a rent-division problem with $R = 100$

	a	b
u_{Alice} :	α	$100 - \alpha$
u_{Bob} :	β	$100 - \beta$

Can you find h_i such that $t_{Alice} + t_{Bob} = 1$?

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Can you find h_i such that $t_{Alice} + t_{Bob} = 1$?

Conclusion: the main problem with VCG is that it is not budget-balanced. You either need to burn money or to add them to the system.

Mechanism of Francis Su

A rent-division mechanism of Francis Su[3] [4]

- **Minimal assumptions on preferences.** It is assumed that for any vector of prices p any agent i can tell his most preferred room. The only condition on preferences is:
  - *Miserly tenants:* every agent prefers a free room (a room with a rent of 0) over any other room.
- **Interactive elicitation procedure.** Reporting general preferences is impossible $\Rightarrow$ mechanism learns only partial information about preferences asking questions like "which room do you prefer at this price?" in an interactive way.

  Remark: interactive procedures are also used in multi-unit auctions
- **Computes approximately Envy-Free allocation.** Given $\epsilon > 0$ finds such an allocation (σ, p) that for any agent i is is enough to change the price-vector by at most ε to make him non-envious.

[3] Francis Edward Su. Rental harmony: Sperner's lemma in fair division. Amer. Math. Monthly, 106(10):930–942, 1999.
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How it works? See the whiteboard:

- The set of prices is an interval $p_a + p_b = R$, $p_a, p_b \geq 0$
- Miserly tenants $\Rightarrow$ when $p_a = 0$ both agents prefer a, the same for $p_b = 0$
- Cut the interval into ε-baby-intervals
- At even endpoints ask Alice what is her preferred room at this price? At odd endpoints ask Bob.
- This produces a labeling of endpoints by a, b with the leftmost label a and the rightmost b
- Starting from the leftmost label go to the right until we find a baby-interval with two different labels.
- Give each agent what is written on his label and select price from the baby-interval. We are done!

Three and more agents: The same ideology but now the set of prices is a triangle $p_a + p_b + p_c = R$, $p_a, p_b, p_c \geq 0$, and existence of a baby-triangle with three different labels is a version of the Sperner Lemma.

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