Introduction to modern Fair Division. Appeal and challenges of the competitive approach

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Economic design & the computer revolution

The problem of Economic Design: How to design the rules of interaction to achieve the desired outcome?

The time-line:

- before mid-nineties: theoretical area at the interface of GT and Microeconomics
- since mid-nineties: computers and the Internet ⇒ practical implementation of theor results
    - 1990s: Auctions
    - 2000s: Large centralized markets (job markets, school choice, transplants)
    - 2010s: Fair Division and other mechanisms on micro-level (dividing the rent, peer grading, computational resources etc.)

Real-world implementation ⇒ new problems ⇒ new research areas
Algorithmic Game Theory and Computational Social Choice

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The difficulties of the classical approach

- **The classical approach** often leads to impossibility results. Famous examples:

  Arrow's impossibility & the Gibbard-Satterthwaite impossibility theorems
- **Algo GT** focuses on positive results. Tools for avoiding impossibilities:
  - average-case instead of the worst-case
  - approximate requirements
  - quantitative analysis instead of qualitative (if the requirement is violated, quantify by how much)
- **The classical approach** assumes that agents can formulate and report very complex preferences. Example:
  - general preferences on bundles of 20 indivisible goods ⇔ list with more than 1000000 items.
- **Algo GT:** complexity issues are important
  - agents have bounded cognitive abilities ⇒ importance of simple preference domains
  - the outcome of a mechanism has to be efficiently computable

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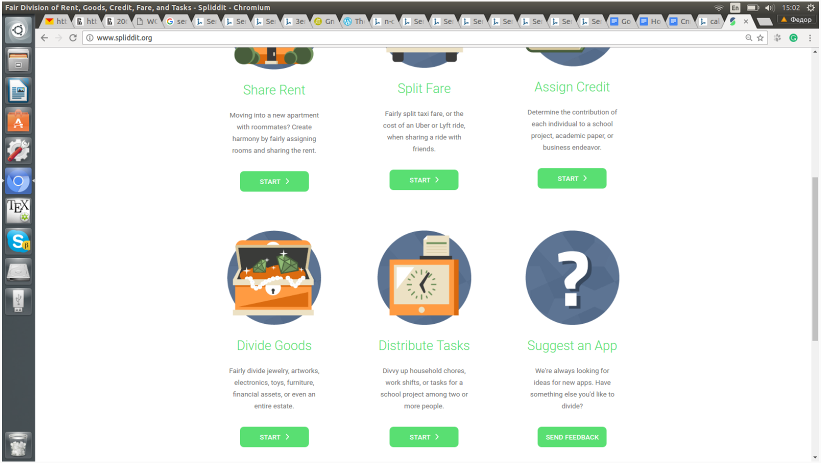
Today

We will illustrate the interplay of the classic and modern approaches to Fair Division mechanisms without monetary transfers.

- Examples: division of a common property (partners dissolving their partnership, divorce, inheritance), charity, seats in overdemanded courses, resources within the firm (office space, IT facilities, bonuses), computational resources in a network, bandwidth among mobile phones

Let's look how it works on `Spliddit.org`, an online fair-division platform launched by the team of Ariel Procaccia (Carnegie Mellon University)

Fair division of private goods on Spliddit.org



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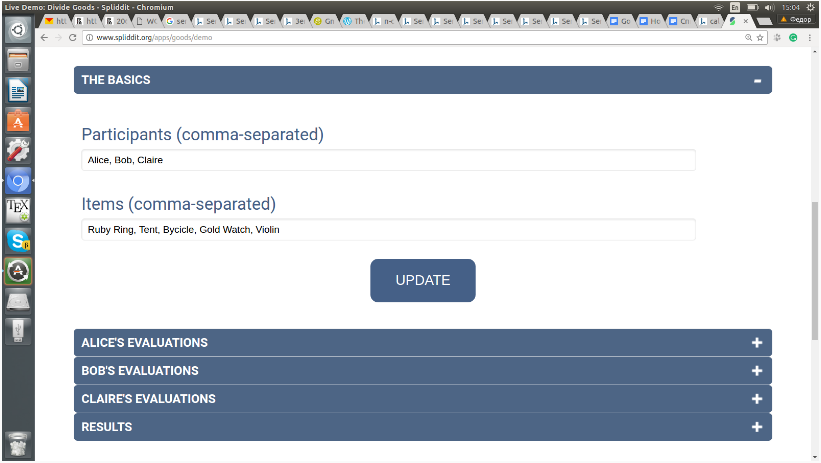

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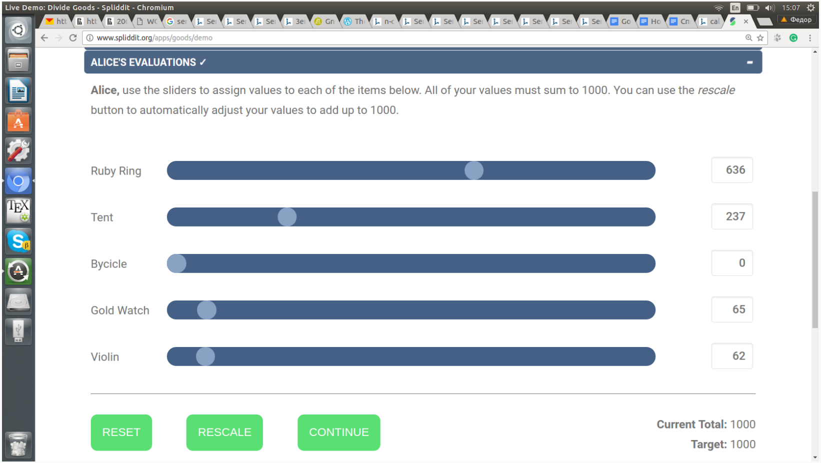

- Each agent redistributes 1000 points among goods, and these reported values reflect "importance" of a good to an agent.

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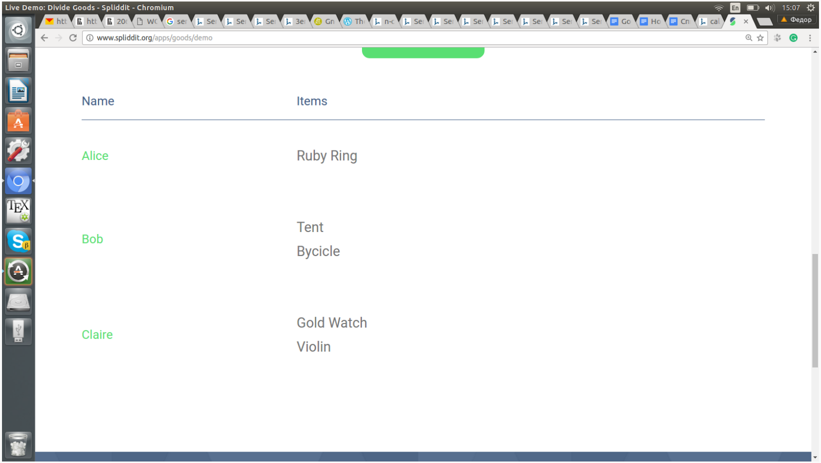

- This mechanism assumes that goods are indivisible and will be described in the next lecture of Vasilis Gkatzelis. We consider a simpler case of divisible items.

Outline:

Lecture 1. Divisible items

- Fairness. Mathematicians cut a cake
- Fairness & Efficiency. Microeconomists divide divisible private goods
  - Examples: Utilitarian, Egalitarian, and the Nash rule
- The Competitive Approach
  - Relation to the Nash Rule
  - What if we divide bads, not goods? [1]

Lecture 2 (Vasilis Gkatzelis). Strategic issues & Indivisible items

[1]based on joint papers with Anna Bogomolnaia, Hervé Moulin, Elena Yanovskaya
"Competitive division of a mixed manna", Econometrica. 2017. V.85:6. P.1847-1871
"Dividing goods *or* bads under additive utilities", SocChoice&Welfare . 2018. P. 1-23
"Dividing goods *and* bads under additive utilities" arXiv:1610.03745 [cs.GT]

Fairness. Mathematicians cut a cake

Cake-cutting problem

The first rigorous result on Fair Division:
Steinhaus, H. (1948). The problem of fair division. Econometrica, 16, 101-104.

The problem:

- A divisible non-homogeneous resource $G = [0, 1]$ (land, CPU time, pizza with different toppings) is to be divided among a finite set of agents $N = \{1, 2..n\}$
- Agent i has a utility function u_i on subsets of G. It is
  - additive ($u_i(B \cup B') = u_i(B) + u_i(B')$ for $B \cap B' = \emptyset$)
  - normalized $u_i(G) = 1$
  - non-atomic $u_i(\{x\}) = 0$ $\forall x \in G$.
- **The goal:** find a "fair" division of the cake: $G = G_1 \sqcup G_2 \sqcup ..G_n$

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A well-known protocol

"Cut and chose"

Two agents. Agent 1 cuts a cake into two pieces B, B', equal from his point of view: $u_1(B) = u_1(B')$. Agent 2 takes the most preferred piece, agent 1 gets the remaining.

In what sense is it fair?

Fairness. The two dominant criteria:

Envy-Freeness

Every agent prefers his allocation to the allocation of any other agent:

$$u_i(G_i) \geq u_i(G_j) \text{ for all } i, j \in N.$$

Fair Share Guaranteed (aka Proportionality or Equal Division Lower Bound)

Every agent prefers his allocation the "equal division":

$$u_i(G_i) \geq 1/|N|$$

What is stronger, E-F or FSG?

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What is stronger, E-F or FSG? For $|N| \geq 2$ E-F implies FSG. For $|N| = 2$ E-F $\Leftrightarrow$ FSG.

Evolution of cake-cutting[2]

- Dubins-Spanier (1961) "moving-knife procedure": FSG for $|N| > 2$
- Selfridge-Conway (1961): E-F for $|N| = 3$ with 9 cuts
- Brams-Taylor (1995): E-F for $|N| > 3$ with unbounded number of cuts
- bounded E-F protocols: $|N| = 4$ Brams-Taylor-Zwicker (1997), $|N| = 5$ Saberi-Wang (2009)
- Aziz-Mackenzie (2016): bounded E-F procedure with "at most" $n^{n^{n^{n^{n^{n}}}}}$ cuts
- Computer scientists in last 10 years: complexity of cake-cutting (minimal number of cuts & queries)

[2] Procaccia, Ariel D. Cake cutting: not just child's play. Communications of the ACM 56.7 (2013): 78-87.

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Criticism of cake-cutting

- Not much realistic. Instead of a "cake" we usually have a family of private goods
- **Economists:** Most of the results are focused on fairness without efficiency

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Efficiency. Microeconomists divide divisible private goods

Features:

- Fairness is combined with efficiency
- The model becomes more realistic
  - Examples: inheritance, common property between partners, seats in overdemanded courses, etc
- Wait... Usually the goods are indivisible!

Lifehack: what is 0.3 of a bicycle?

- randomization: getting the bicycle with probability 0.3
- time-sharing: using bicycle 30% of time

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The model

- A finite set of divisible goods $G = \{a, b, c, ..\}$, each in the unit amount, is to be allocated to agents, $N = \{1, 2, 3.., n\}$ without money transfers
- $z_i = (z_{ia}, z_{ib}, z_{ic}..) \in R_+^G$ is a bundle of goods received by agent i
- an allocation $z = (z_i)_{i \in N}$ is a collection of bundles $z_i \in \mathbb{R}_+^G$ of all agents with the condition that all goods are distributed: $\forall g \in G \ \sum_{i \in N} z_{ig} = 1$
- Preferences are given by utility functions: $u_i(z_i)$ is agent i's utility
    - Classic approach: mild assumptions on u_i, e.g., general Arrow-Debreu domain of preferences
    - Algo GT approach: u_i has to be easy to formulate and to report, e.g.,
        - additive utilities (no complementarities): $u_i(z_i) = \sum_{g \in G} u_{ig} z_{ig}$
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We assume **additive utilities** and normalization: $\sum_{g \in G} u_{ig} = 1$ (or 100 or 1000 like on Spliddit)

The objectives

- **Fairness:** Envy-Freeness versus Fair Share Guaranteed:

$$u_i(z_i) \geq u_i(z_j)\ \forall i, j \in N \quad \text{versus} \quad u_i(z_i) \geq \frac{1}{|N|}$$

- **Efficiency (Pareto-optimality):** An allocation z is Efficient iff there is no allocation z' weakly preferred by all agents and by at least one strictly.

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$$u_i(z_i) \geq u_i(z_j)\ \forall i, j \in N \quad \text{versus} \quad u_i(z_i) \geq \frac{1}{|N|}$$

- **Efficiency (Pareto-optimality):** An allocation z is Efficient iff there is no allocation z' weakly preferred by all agents and by at least one strictly.

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Question: Is fairness compatible with efficiency? We will see soon.

Example: the Utilitarian rule

$$z: \quad \sum_{i \in N} u_i(z_i) \to \max$$

Efficient but very unfair. Example:

	sangria	*spritz*	*wine*
u_{Alice} :	80	10	10
u_{Bob} :	10	80	10
u_{Claire} :	10	10	80
u_{Dave} :	33	33	34

$\Longrightarrow$

	sangria	*spritz*	*wine*
z_{Alice} :	1	0	0
z_{Bob} :	0	1	0
z_{Claire} :	0	0	1
z_{Dave} :	0	0	0

Flexible agents may get nothing!

Example: the Egalitarian rule

$$z: \quad \min_{i \in N} u_i(z_i) \to \max$$

- introduced by Pazner and Schmeidler [3]

Properties:

- Efficient (need leximin extension)
- FSG?
- E-F?

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- FSG? Yes, because equal division ($z_{ig} = \frac{1}{|N|}$) is a feasible allocation.
- E-F? For $|N| > 2$, no. Example:

$$\begin{array}{rcc} & a & b \\ u_{Alice}: & 6 & 6 \\ u_{Bob}: & 8 & 4 \\ u_{Claire}: & 9 & 3 \end{array} \implies \begin{array}{rcc} & a & b \\ z_{Alice}: & 0 & \frac{18}{23} \\ z_{Bob}: & \frac{11}{23} & \frac{5}{23} \\ z_{Claire}: & \frac{12}{23} & 0 \end{array}, \quad \text{Claire envies Bob.}$$

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Example: the Nash rule aka MaxNashProduct

A compromise between Utilitarian and Egalitarian approaches:

$$z: \quad \mathcal{N}(z) = \prod_{i \in N} u_i(z_i) \to \max$$

a similar rule was introduced by J. Nash in the context of axiomatic bargaining[4]

Properties:

- Efficiency
- FSG? Yes!
- Envy-Freeness? Yes!
- A convex optimization problem ⇒ uniqueness, robustness, can be approximately computed by standard gradient methods, or exactly using primal-dual algorithms

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- FSG? Yes!
- Envy-Freeness? Yes!
  Proof: For a maximizer z and a pair of agents i, j consider $z(\varepsilon)$ where $z_i(\varepsilon) = z_i + \varepsilon z_j$ and $z_j(\varepsilon) = (1 - \varepsilon) z_j$. Therefore $\log \mathcal{N}(z) \geq \log \mathcal{N}(z(\varepsilon))$ and

$$\frac{\partial \log \mathcal{N}(z(\varepsilon))}{\partial \varepsilon} \leq 0 \Leftrightarrow \frac{u_i(z_j)}{u_i(z_i)} - 1 \leq 0 \Leftrightarrow u_i(z_i) \geq u_i(z_j). \quad \square$$

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[4] Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V. V. (Eds.). (2007). Algorithmic game theory (Vol. 1). Cambridge: Cambridge University Press. **Chapters 5,6**

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There are many confirmations that the Nash rule is **the best rule to divide goods under additive utilities**. Why is the Nash product so specific?

The Competitive Approach

Envy-freeness as equal choice opportunities

Fairness as equal choice opportunities

Carol likes candies and Bob likes beer; each of them spends 100 euros in a supermarket on their favorite products. Will they envy each other?

No, because both select the best bundle of goods from the same choice set (their budget set).

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Remark: equal-opportunity approach can be extended beyond budget sets and fair-division applications.[5]

[4]Varian, H. R. (1974). Equity, envy, and efficiency. Journal of economic theory, 9(1), 63-91.

[5]Richter, Michael, and Ariel Rubinstein. Normative Equilibrium: The permissible and the forbidden as devices for bringing order to economic environments. Working paper, 2018

The Competitive Equilibrium with Equal Incomes (CEEI)

Informal definition:

- give every agent a unit amount of "virtual" money
- select prices s.t. the "demand" equalizes "supply": when each agent buys the best bundle he/she can afford, all items are sold and all money spent.

aka Competitive Rule, Pseudo-Market mechanism, Equilibrium of the Fisher Market, or Kelly's proportional fairness.

The resulting allocation is

- envy-free $\Longleftarrow$ equal choice opportunities
- efficient $\Longleftarrow$ "invisible hand" of Adam Smith (1st fundamental theorem of Welfare Economics)

This holds in a very general setup (for Arrow-Debreu preferences).

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Formal definition

z is a competitive allocation if there is a vector of prices $p \in \mathbb{R}_+^G$ such that for any agent $i \in N$

$$z_i \text{ maximizes } u_i \text{ over the budget set } \{y \in \mathbb{R}_+^G : \langle y, p \rangle \leq 1\}.$$

Properties: Envy-Freeness & Efficiency, Existence (non-constructive fixed-point arguments based on the Kakutani theorem), but...

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Theorem (Eisenberg (1961), Gale (1960))

For homogeneous utilities (i.e. $u_i(\lambda y) = \lambda u_i(y), \quad \forall \lambda > 0$, e.g, additive or Leontief) **CEEI coincides with the Nash rule.**

Chipman's[6,7] proof of Eisenberg-Gale theorem

First check that competitive z with price vector p maximizes $\mathcal{N}$:

- z_i maximizes $u_i(y_i)$ over bundles $y_i \in \mathbb{R}_+^G$ with price $\langle y_i, p \rangle = 1 \Longrightarrow$

$$z \in \operatorname{argmax}_{\substack{y = (y_i)_{i \in N}: \\ y_i \in \mathbb{R}_+^G \\ \langle y_i, p \rangle = 1}} \prod_{i \in N} u_i(y_i) \quad \left[\text{no feasibility constraint } \sum_i y_i = 1!\right]$$

[6] J. S. Chipman. 1974. Homothetic preferences and aggregation, *Journal of Economic Theory*, 8, 26-38.

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- Instead of $\langle y_i, p\rangle = 1$ for all i, we can write "aggregate budget constraint" $\langle \sum_{i\in N} y_i, p\rangle = |N|$. Follows from **budget equalizing trick**: if a maximizer $y = (y_i)_{i\in N}$ has unequal budgets $b_i = \langle y_i, p\rangle$, then by defining $y_i' = \frac{y_i}{b_i}$ we increase the product. Indeed

$$\prod_{i\in N} \frac{1}{b_i} \geq 1 \quad \text{for} \quad \sum_{i\in N} b_i = |N|.$$

- The set $\{y = (y_i)_{i\in N} : \ y_i \in \mathbb{R}_+^G, \ \langle \sum_{i\in N} y_i, p\rangle = |N|\}$ contains all feasible allocations. Thus z maximizes $\mathcal{N}$ over feasible allocations.

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By uniqueness of $\mathcal{N}$-maximizers, any maximizer is competitive. □

What if we divide bads, not goods?[6]

[6] based on
Anna Bogomolnaia, Hervé Moulin, Fedor Sandomirskiy, Elena Yanovskaya
"Competitive division of a mixed manna", Econometrica. 2017. V.85:6. P.1847-1871
"Dividing goods *or* bads under additive utilities", SocChoice&Welfare . 2018. P. 1-23
"Dividing goods *and* bads under additive utilities" arXiv:1610.03745 [cs.GT]

Motivation

- Most of the results in fair division are about goods
    - Exception: F. Su. (2002, 2009), burnt cake cutting
- But many real problems involve bads
    - e.g., house chores, teaching loads, noxious facilities
- Or goods and bads at the same time

Why goods $\neq$ bads? Turning bads into goods.

Bads instead of goods: $u_{ib} \leq 0$ for all agents and items.

Toy example

- 4 agents divide 1 hour of painful work b
- introduce auxiliary good "not doing b"
- we have 3 hours of "not doing b" to distribute, but no agent can consume more than one hour.

Corollary: A problem with bads can be reduced to a constrained problem with goods.

Why goods $\neq$ bads? Extension of the Nash rule to bads, failed attempts

Ideas:

- **Minimize the product of disutilities** $\mathcal{N}(z) = \prod_{i \in N} |u_i(z_i)|$
  Very unfair: picks an allocation with $\mathcal{N}(z) = 0$ that gives no bads to one of agents
- **Maximize the product of disutilities**
  Inefficient: is dominated by equal division $z_{ib} = \frac{1}{|N|}$

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To extend the Nash rule to bads we need its connection to **the CEEI**

CEEI for a mixture of divisible goods & bads

- We assume that utilities are homogeneous (e.g., additive, Leontief)

How to define CEEI?

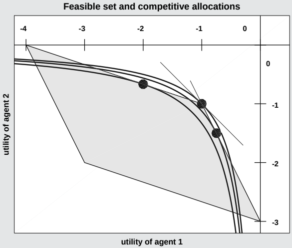
- Allow prices and budgets of both signs.

Competitive allocations exist (follows from Mas-Colell, 1982), are envy-free and efficient.

Analog of Eisenberg-Gale theorem

Geometry of CEEI for bads

CEEI are critical points (local minima, local maxima, and saddle points) of the Nash product on the efficient frontier.



Thus finding critical points is **not a convex** optimization problem[7].

The result extends to **mixture of goods and bads.** The proof is based on an extension of Chipman's demand-aggregation ideas (difficulty: cannot use uniqueness anymore).

[7] Branzei, Sandomirskiy (2019) "Competitive division of chores" show that for fixed N, still all competitive allocations can be computed in polynomial time

New issue: multiplicity

- There are problems with exponential number of CEEI if $|N|$ (the number of agents) $\asymp |G|$ (the number of items)
- Multiplicity $\Rightarrow$ additional negotiations what outcome to chose $\Rightarrow$ need to find a good single-valued selection
- There are no continuous selections & there are no other single-valued, efficient, and envy-free division rules that are continuous.

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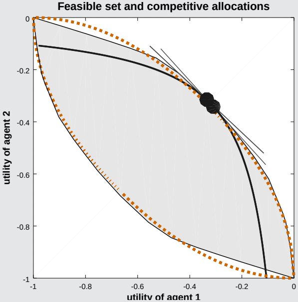
Multiplicity disappears for typical large problems

- u_{ib} are i.i.d. random variables uniformly distributed on $[-\frac{1}{m}, 0]$.

Proposition

Two agents divide m bads, $m \to \infty$. Fix $\varepsilon > 0$. Utility vectors of all competitive allocations are concentrated in ε-neighborhood of $\left(-\frac{1}{3}, -\frac{1}{3}\right)$ with probability $p_m \to 1$.



Example with 15 bads.

Summary

- Modern approach to fair division inspired by computer scientists: narrow preference domain, focus on positive results implementable in practice
- CEEI
  - CEEI provides a general methodology to design fair and efficient rules
  - CEEI is defined implicitly $\Rightarrow$ non-transparent and leads to algorithmic difficulties
  - Difficulties disappear for goods under homogeneous utilities: CEEI=MaxNashProduct $\Rightarrow$ uniqueness and convexity of optimization problem
- Bads$\neq$goods: multiplicity of CEEI $\Rightarrow$ algorithmic difficulties, the question of choosing a single-valued selection
- Connection to the Nash product is useful when looking for an extension of CEEI to new classes of problems. For indivisibilities, depending on the language (Nash product or CEEI) one gets different extensions but this is from the lecture of Vasilis.

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  - Difficulties disappear for goods under homogeneous utilities: CEEI=MaxNashProduct $\Rightarrow$ uniqueness and convexity of optimization problem
- Bads$\neq$goods: multiplicity of CEEI $\Rightarrow$ algorithmic difficulties, the question of choosing a single-valued selection
- Connection to the Nash product is useful when looking for an extension of CEEI to new classes of problems. For indivisibilities, depending on the language (Nash product or CEEI) one gets different extensions but this is from the lecture of Vasilis.

Summary

- Modern approach to fair division inspired by computer scientists: narrow preference domain, focus on positive results implementable in practice
- CEEI
  - CEEI provides a general methodology to design fair and efficient rules
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What to read?

Surveys:

- **Economist's view on the current trends in fair-division:**
  Herve Moulin Fair Division in the Internet Age. Annual Review of Economics 11 (2019).
- **Classic microeconomic approach to fair-division:**
  William Thomson "Fair Allocation Rules" Working Paper No. 539, Rochester University, December 2007
- **Computer-scientist's view on cake-division:**
  Procaccia, Ariel D. Cake cutting: not just child's play. Communications of the ACM 56.7 (2013): 78-87.

Big books:

- **Algorithmic Game Theory and Computational Social Choice (with chapters on fair division):**
Brandt, F., Conitzer, V., Endriss, U., Procaccia, A. D., & Lang, J. (Eds.). (2016). Handbook of computational social choice. Cambridge University Press.
Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V. V. (Eds.). (2007). Algorithmic game theory (Vol. 1). Cambridge: Cambridge University Press.
- **Classic economic book:**
Heeve Moulin Fair division and collective welfare. – MIT press, 2004.
- **Introductory level (game theory, mechanism design including FD):**
Karlin, Anna R., and Yuval Peres. Game theory, alive. Vol. 101. American Mathematical Soc., 2017.
Shoham, Y., & Leyton-Brown, K. (2008). Multiagent systems: Algorithmic, game-theoretic, and logical foundations. Cambridge University Press.