Methods of Optimal Transportation in Bayesian Persuasion & Auctions

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What will we see?

Economic applications of **non-classic transportation problems**

Classic Transportation Problem

Given: the utility function $u: [0,1]^2 \to \mathbb{R}$, marginals $\mu_1, \mu_2 \in \Delta([0,1])$
Find:

$$T_u(\mu_1, \mu_2) = \max_{\substack{\mu \in \Delta([0,1]^2) \\ \text{with marginals } \mu_i}} \int u(x_1, x_2) d\mu(x_1, x_2).$$

Non-classic problems:

- **free marginals:** μ_i are not fixed but must satisfy certain constraints
- **multi-marginal problems**

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What applications will we discuss?

- **Bayesian persuasion:** the key model of strategic communication
  - standard setting has 1 receiver
  - ≥ 2 receivers $\rightarrow$ optimal transport[1]
- **Optimal multi-good auctions:** how to optimally sell *m* goods to *n* buyers with i.i.d. values?[2]

[1] Arieli, I., Babichenko, Y., Sandomirskiy, F., & Tamuz, O. (2020) Feasible Joint Posterior Beliefs

[2] C.Daskalakis, A.Deckelbaum, C.Tzamos (2017) Strong Duality for a Multiple-Good Monopolist Econometrica

Bayesian persuasion

Bayesian persuasion (aka Information Design)

The question:

How to induce the desired behavior of a decision-maker by changing the information available to him?

- A young field. The origin:

Bayesian persuasion

E Kamenica, M Gentzkow - American Economic Review, 2011 - aeaweb.org

When is it possible for one person to persuade another to change her action? We consider a symmetric information model where a sender chooses a signal to reveal to a receiver, who then takes a noncontractible action that affects the welfare of both players. We derive ...

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- Popularity: often explicit solutions, many applications[3]

[3]E. Kamenica (2019) Bayesian persuasion and information design Annual Review of Economics

Toy example: a court problem

- 75% of defendants are innocent ($\theta = 0$), 25% are guilty ($\theta = 1$)
- Prosecutor (P) observes θ, Judge (J) does not
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What should P do?

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- Reveal $\theta \Longrightarrow$ 25% are convicted
- Send a signal $s \in S$ with θ-dependent probabilities $\pi_\theta \in \Delta(S)$:
  - J's posterior $x = \mathbb{P}(\theta = 1 \mid s)$ and $\begin{cases} \text{convicts} & x \geq 0.5 \\ \text{acquits} & x < 0.5. \end{cases}$
  - P's problem.

$$\text{maximize} \quad \mathbb{E}\left[\mathbb{1}_{x \geq 0.5}\right] \quad \text{over signlling policies } (S, \pi)$$

  - The optimum:

	$S = \{$ "maybe innocent",	"guilty" $\}$
$\pi_{\theta=0}$	1	0
$\pi_{\theta=1}$	$\frac{1}{3}$	$\frac{2}{3}$

Convicts 50%!

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Some other applications:

- Employers and universities: $\theta =$quality of a student (good/bad), U wants a good placement for any student, E wants good candidates.
  - Explains coarse grading in schools, universities, and industries:[4]
    "When recruiters call me up and ask me for the three best people, I tell them, "No! I will give you the names of the six best."

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  - Explains why you cannot order the apts by rating or price on AirBNB[5]

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- Police & drivers: $\theta =$ whether the region is patrolled (yes/no). P wants D to obey the speed limit, D wants to obey only if the region is patrolled.

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The classic model with 1 receiver

- A random state $\theta \in \{0,1\}$ with prior probability $p = \mathbb{P}(\theta = 1)$
- **Definition:** A distribution $\mu \in \Delta([0,1])$ is a feasible distribution of posteriors if there exists[6] a sigma-field[7] $\mathcal{F}$ such that $\mathbb{P}(\theta = 1 \mid \mathcal{F})$ has distribution μ.

Persuasion problem

Given: prior p and utility $u = u(x)$

Find:

$$V(p) = \max_{\text{feasible } \mu \in \Delta([0,1])} \int_{[0,1]} u(x) d\mu(x)$$

[6] The probability space must be rich enough, say $[0,1]$ with the Lebesgue measure.
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$\mathrm{Cav}\,[u]$**-theorem (Aumann & Maschler, 60ies)**

$$V(p) = \mathrm{Cav}\,[u](p), \quad \text{where } \mathrm{Cav}\,[u] = \min_{\substack{\text{concave } f: \\ f \geq u}} f$$

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Proof:

"$\leq$:" $u \leq \mathrm{Cav}\,[u] \Rightarrow V \leq \mathrm{Cav}\,[u]$ by Jensen's inequality

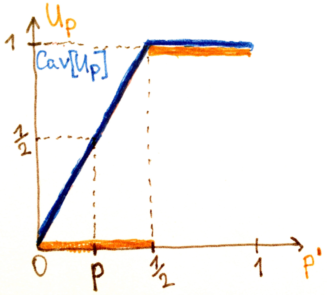
"$\geq$": $V \geq u$, V is concave $\Rightarrow V \geq \mathrm{Cav}\,[u]$. □

The classic model with 1 receiver

Example: back to the court

$$p = 0.25 \text{ and } u(x) = \mathbb{1}_{x \geq 0.5}$$

The function u and its concavification:



The optimal $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{2}}$.

$n \geq 2$ receivers[8]

- $\theta \in \{0, 1\}$ with prior probability $p = \mathbb{P}(\theta = 1)$
- **Definition:** $\mu \in \Delta([0,1]^n)$ is feasible $\iff \exists$ sigma-fields $\mathcal{F}_1, \dots \mathcal{F}_n$ such that the vector of posteriors $x = (x_1, \dots x_n) \sim \mu$, where $x_i = \mathbb{P}(\theta = 1 \mid \mathcal{F}_i)$.

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Examples with $n = 2$:

- creating discord $u = |x_1 - x_2|^\alpha$
- minimizing covariance $u = -(x_1 - p)(x_2 - p)$

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$n \geq 2$ receivers: criterion of feasibility

- For $\mu \in [0,1]^n$ denote the marginals by $\mu_1, \ldots, \mu_n$
- The martingale property

$$\int_{[0,1]} x_i d\mu_i(x_i) = p, \quad \forall i = 1, \ldots n$$

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$$\mu = (1-p) \cdot \nu^0 + p \cdot \nu^1 \qquad \text{and} \qquad \frac{d\nu_i^1(x_i)}{d\nu_i^0(x_i)} = \frac{x_i}{1-x_i}, \quad \forall i = 1, \ldots n$$

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Proof: let ν^0 and ν^1 be the conditional distributions of $(x_1, \ldots x_n)$ given $\theta = 0$ or $\theta = 1$, respectively.

$n \geq 2$ receivers: persuasion as transportation

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Conclusion

$$V(p) = \max_{\text{marginals } \nu_i^\theta \,:\, (\bigstar) \text{ holds}} \left[(1-p) T(\nu_1^0, \nu_2^0) + p \cdot T(\nu_1^1, \nu_2^1)\right].$$

$n = 2$ receivers: some explicit solutions for $p = \frac{1}{2}$

- $u = |x_1 - x_2|^\alpha$ with $\alpha \in (0, 2]$.

$n = 2$ receivers: some explicit solutions for $p = \frac{1}{2}$

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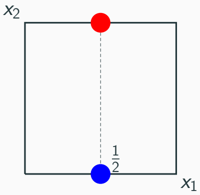

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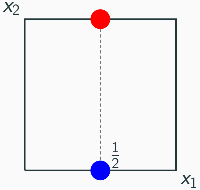

- $\min \mathrm{Cov}(x_1, x_2) =$????

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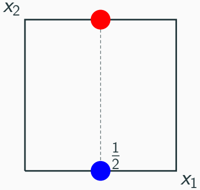
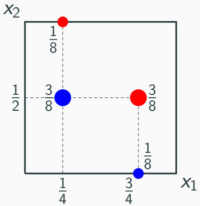


- min $\mathrm{Cov}(x_1, x_2) = -\frac{1}{32}$. Optimal μ:



$n \geq 2$ receivers: how to solve?

Each approach works for u from the last slide:

- **Direct approach**
- **Dual approach**
- **Hilbert-space approach** (in the paper)

$n \geq 2$ receivers: how to solve?

Each approach works for u from the last slide:

- **Direct approach**
  - For $n = 2$ with quadratic $u(x_1, x_2)$, the transportation problem has explicit solutions: anti-monotone coupling
  - Maximization over marginals = an exercise in the calculus of variations
- **Dual approach**
- **Hilbert-space approach** (in the paper)

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An analog of Kantorovich-Rubinstein duality:

$$\begin{aligned} V(p) &= \min_{\text{functions } (f_i)_{i=1\dots n}} \left[(1-p) \cdot \max_x \left(u(x) + \sum_{i=1}^{n} x_i \cdot f_i(x_i) \right) + \right. \\ &+ \left. p \cdot \max_x \left(u(x) - \sum_{i=1}^{n} (1-x_i) f_i(x_i) \right) \right] \end{aligned}$$

Guess primal and dual solutions: zero gap ensures optimality.

- **Hilbert-space approach** (in the paper)

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Open question: Anything beyond quadratic u? Other sources of explicit solutions?

$n \geq 2$ receivers: general property of solutions

- Persuasion problem is an infinite-dimensional LP:
  maximization of a linear functional over a convex set of feasible distributions μ
- **Bauer's principle:** optimum is at an extreme points

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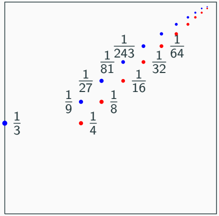
- There are extreme μ with countable support:



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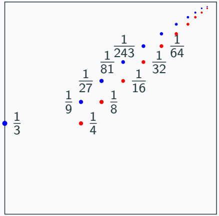

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Question: Non-atomic extreme μ?

Optimal way to sell multiple goods

The model

- n agents, m goods
- values $v_{i,j}$ are i.i.d. with density f

How to maximize revenue from selling? Assumptions:

- f is known, realizations of $v_{i,j}$ are not
- each agent acts in his best interests

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What is known?

- $n \geq 2, m = 1$ **(the classic auction theory)**: everything
- $n = 1, m \geq 2$ **(selling many goods to one agent)**:
    - optimal mechanisms in particular cases
    - connections to optimal transport
- $n \geq 2, m \geq 2$ **(auctions with multiple goods)**: nothing

Warm-up: $n = m = 1$

- How to sell one good to one agent with the value $v \sim f(v)dv$?

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Theorem (Myerson (1981))

Take it or leave it with p^* is the optimal mechanism

$m \geq 2$ goods, $n = 1$ agent: optimal mechanisms

- the agent has i.i.d. values $v = (v_1, \ldots, v_m) \sim f(v)dv$
- if the agent gets the bundle of goods $x = (x_1, \ldots, x_m) \in [0,1]^m$ for price p, his utility is $\langle x, v \rangle - p$
- Is selling each good separately always optimal?
- Is bundling all goods together always optimal?
- Is $x \in \{0,1\}^m$ enough?
- **menu mechanism:** chose the best option from the menu
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Revelation principle

Any mechanism is equivalent to a menu mechanism.

$m \geq 2$ goods, $n = 1$ agent: finding optimal menus

- the menu $M \subset \mathbb{R}_+ \times [0,1]^m$
- utility obtained by an agent with values $v = (v_1, \ldots, v_m)$:

$$u_M(v) = \max_{(p,x) \in M} \langle x, v \rangle - p,$$

- u_M is convex and

$$x(v) = \partial u_M(v), \quad p(v) = u_M(v) - \langle x(v), v \rangle$$

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Theorem (Rochet and Chone (1998))

$M \leftrightarrow u_M$ is a bijection between menus and convex u_M with $u_M(0) = 0$ and $\partial u_M \in [0,1]^m$.

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[integrating by parts]

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[integrating by parts]

$$= \max_{\substack{\text{convex } u \\ u(0)=0,\ \partial u \in [0,1]^m}} \int_{\mathbb{R}^m_+} u(v) d\psi,$$

where $d\psi = \Big((m+1) f(v) + \sum_{j=1}^m v_i \partial_{v_i} f \Big) dv$ (not necessary positive!)

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What is the dual problem?

Definition: 2nd-order stochastic dominance

$$\phi \succ_{SD} \nu \iff \int g d\phi \geq \int g d\nu \text{ for any convex monotone g}$$

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Explicit solutions for $m = 2$:

- **Uniform on $[0, 1]$:** each good for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$
- **Exponential:** sell the goods only together
- **Beta distribution** $Cv^{\alpha-1}(1-v)^{\beta-1}dv$**:** continual menu!!!

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Definition: 2nd-order stochastic dominance

$$\phi \succ_{SD} \nu \iff \int g d\phi \geq \int g d\nu \text{ for any convex monotone g}$$

Theorem (Daskalakis et al (2017))

$$R_m(f) = \min_{\substack{\text{positive measures } \mu \\ \text{on } \mathbb{R}_+^m \times \mathbb{R}_+^m \\ \mu_1 - \mu_2 \succ_{SD} \psi}} \int_{\mathbb{R}_+^m \times \mathbb{R}_+^m} \|v - v'\|_1 d\mu(v, v')$$

Explicit solutions for $m = 2$:

- **Uniform on $[0, 1]$:** each good for $\frac{2}{3}$ or both for $\frac{4-\sqrt{2}}{3}$
- **Exponential:** sell the goods only together
- **Beta distribution $Cv^{\alpha-1}(1 - v)^{\beta-1}dv$:** continual menu!!!

$m \geq 2$ goods, $n \geq 2$ agents?!?

Open problem: Optimal mechanisms for $n, m \geq 2$?

- Even $m = n = 2$ with i.i.d. uniform values is open.

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- Border's theorem[9] reduces the question to 1-agent mechanisms.
- As before: 1-agent mechanisms $\leftrightarrow$ convex u
- Border's theorem $\rightarrow$ new constraint on u subsuming $\partial u \in [0,1]^m$:

$$\partial_{v_j} u(v) \prec_{SD} \xi^{n-1} \quad \forall j = 1, \dots m,$$

where v is random with density f and ξ is uniform on $[0,1]$.

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Corollary:

$$R_{n,m}(f) = \max_{\substack{\text{convex monotone } u \\ u(0)=0,\ \partial_{v_j} u(v) \prec_{SD} \xi^{n-1} \forall j}} n \cdot \int_{\mathbb{R}_+^m} \Big(u(v) - \langle \partial u(v), v \rangle \Big) f(v) dv.$$

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Question: Any explicit solutions? Any handy dual?

The end

Applications we haven't talked about:

- **Robustness of probabilistic models w.r.t. prior distribution:** Kantorovich metric (aka Wasserstein or earth-mover distance)
- **Allocation markets with transferable utility** (Shapley-Scarf): maximal-welfare matchings are the solutions to optimal transport
- **Repeated games with incomplete information** lead to multi-marginal martingale transportation problems[9]
- and many others...

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Thank you!

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