Lecture 1: zero-sum games with incomplete information

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Outline:

- Reminder: martingales and posterior probabilities
- Static zero-sum games with incomplete information on one side
- Repeated zero-sum games with incomplete information on one side:
    - $\mathrm{Cav}\,[u]$-theorem via Blackwell's approachability
    - $\mathrm{Cav}\,[u]$-theorem via martingales of posterior beliefs

Reminder: martingales and posterior probabilities

Martingales

probability $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$

Definition

A sequence of random variables $\xi_0, \xi_1, \xi_2, \dots$ is a martingale if ξ_t is $\mathcal{F}_t$-measurable and

$$\mathbb{E}[\xi_{t+1} \mid \mathcal{F}_t] = \xi_t$$

Interpretation: the best prediction of the future value = current value $\Rightarrow$ wide use in models of learning.

Main example: martingale of posteriors

- Unobservable state $\theta \in \{0,\ 1\}$ with prior probability $\mathbb{P}(\theta = 1) = p$.
- An agent sequentially observes signals $s_1, s_2, s_3 \ldots$ which have arbitrary joint distribution with θ.
- The agent computes his posterior probability $p_t = \mathbb{P}[\theta = 1 \mid s_1, s_2, \ldots s_t]$ using the Bayes rule.

Proposition

The sequence $p_0 = p, p_1, p_2, \ldots$ is a martingale with values in $[0, 1]$

Interpretation: best prediction of tomorrow's belief is today's belief $\Leftrightarrow$ rationality property: time-consistency of beliefs.

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Proof: Denote $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t = \Sigma(s_1, s_2, \ldots s_t)$. Then

$$p_t = \mathbb{P}[\theta = 1 \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\theta=1\}} \mid \mathcal{F}_t].$$

By the telescopic property of conditional expectations

$$\mathbb{E}[p_{t+1} \mid \mathcal{F}_t] = \mathbb{E}\Big[\mathbb{E}[\mathbb{1}_{\{\theta=1\}} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t\Big] = \mathbb{E}[\mathbb{1}_{\{\theta=1\}} \mid \mathcal{F}_t] = p_t. \quad \square$$

Static zero-sum games with incomplete information on one side

The model

Static zero-sum game $G(p)$ with one-sided incomplete information

1. the "state of nature" $\theta \in \{0,\ 1\}$ with prior $\mathbb{P}(\theta = 1) = p$ is realized.
   - Player 1 observes θ
   - Player 2 observes nothing but knows p
2. Players play a zero-sum game with $n \times m$ payoff matrix $A^\theta = (A^\theta_{i,j})_{i \in [n], j \in [m]}$ which depends on θ.

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- Player 1 specifies $x = (x^0, x^1)$, where $x^\theta \in \Delta_n$
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The payoff to Player 1

$$\mathbb{E}_{\theta, i \sim x^\theta, j \sim y}\left[A^\theta_{i,j}\right] = (1-p) \cdot \sum_{i,j} x_i^0 A^0_{i,j} y_j + p \cdot \sum_{i,j} x_i^1 A^1_{i,j} y_j$$

= $(-1)\cdot$**payoff to Player 2**

The value

P1 can guarantee: $\max_x \min_y \left[(1-p) \cdot \sum_{i,j} x_i^0 A_{i,j}^0 y_j + p \cdot \sum_{i,j} x_i^1 A_{i,j}^1 y_j\right]$

P2 can defend: $\min_y \max_x \left[(1-p) \cdot \sum_{i,j} x_i^0 A_{i,j}^0 y_j + p \cdot \sum_{i,j} x_i^1 A_{i,j}^1 y_j\right]$

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$$V(p) = \max_x \min_y \left[(1-p)\cdot\sum_{i,j} x_i^0 A_{i,j}^0 y_j + p\cdot\sum_{i,j} x_i^1 A_{i,j}^1 y_j\right] = \min_y \max_x$$

Question: $\max\min = \min\max$ for zero-sum games with complete information. Why here?

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- **Answer 1:** Sets of strategies are convex and compact, the payoff is affine in strategies of each player $\Rightarrow$ apply the min-max theorem.
- **Answer 2:** Reduce $G(p)$ to a matrix game with complete information:
    - pure strategy of Player 1 is a function $i' : \quad \theta \to i^\theta$ (n^2 pure strategies).
    - For a combination of pure strategies: $i' = (i^0, i^1)$ and j the payoff $A'_{i',j} = (1-p)\cdot A^0_{i^0,j} + p \cdot A^1_{i^1,j}$.
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A mystery: The part is bigger than the whole!

Properties of the value

Lemma: concavity and Lipschitz property

$V(p)$ is a concave function of p and $\left|\frac{V(p)-V(p')}{p-p'}\right| \leq 2\max_{i,j,\theta}\left|A^{\theta}_{i,j}\right|$.

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So V is the minimum over y of the family of affine functions. □

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Notation: The value $u(p) = \mathrm{val}[A^{\mathrm{NR}}(p)]$.

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$$V(p) \geq u(p).$$

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Proof: Player 1 "forgets" θ and plays the opt. strategy from $A^{\mathrm{NR}}(p)$.

The $\mathrm{Cav}\,[u]$-lower bound on the value

Concavification: For a continuous function f on a compact convex set

$$\mathrm{Cav}\,[f](y) = \min \{\varphi(y) \,:\, \varphi \text{ is concave and } \varphi(\cdot) \geq f(\cdot)\}\,.$$

So $\mathrm{Cav}\,[f]$ is the minimal concave function dominating f.

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Theorem (R.Aumann, M.Maschler, 1960ies)

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Proof: $V \geq u$ and V is concave. □

Example

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

1. Find the value and optimal strategies in $G(p)$
2. Find the value of the non-revealing game $A^{\mathrm{NR}}(p)$

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      $$V(p) = \min\{1 - p, p\}.$$

    - Optimal reply is unique $\Rightarrow$ opt. strategy of P2 is playing Right if $p \leq \frac{1}{2}$ and Left for $p \geq \frac{1}{2}$.
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   - $A^{\mathrm{NR}}(p) = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$. No pure-strategy equilibrium for $p \neq 0,\ 1$ $\Rightarrow$ players use both actions.
   - Optimal mixed strategy makes another player indifferent between the two actions: $(1-p) \cdot x_1 = p \cdot x_2$ and $(1-p) \cdot y_1 = p \cdot y_2$.
   - The optimal strategies $x = y = \big(p, (1-p)\big)$. The value is

     $$u(p) = (1-p) \cdot p.$$

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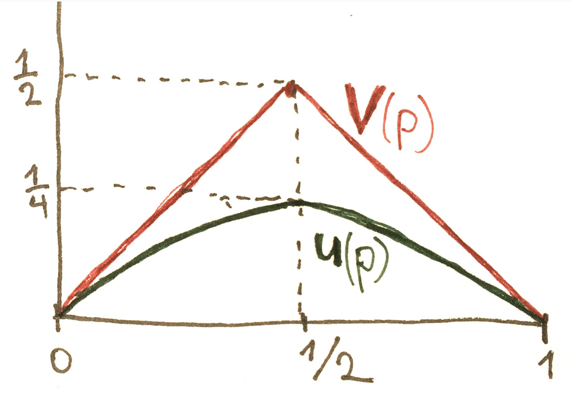

Repeated zero-sum games with incomplete information on one side

Motivation:

Birth in 1960ies: disarmament negotiations US $\leftrightarrow$ USSR. Complex interaction: multistage & both have secrets $\Rightarrow$ interpret the past behavior.

R.Aumann and M.Maschler consulted the US: secret reports[1] ACDA ST/80, ACDA ST/116, ACDA ST/143.

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Static: P1 does not care about revealed information.

Repeated: P2 may guess θ from previous actions of P1 $\Longrightarrow$ P1 balances between using and hiding his information.

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Other examples:

- Nazi's attack to Coventry and broken Enigma cypher (watch "The Imitation Game" about Alan Turing)
- Insider trading on financial markets (Rothschild and Waterloo battle; papers of B. De Meyer)

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The model

T-stage zero-sum game $G_T(p)$ with one-sided incomplete information (RGII)

1. the "state of nature" $\theta \in \{0,\ 1\}$ with prior $\mathbb{P}(\theta = 1) = p$ is realized.
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Behavioral strategies:

- Player 1, for each state θ, time $t = 0, 1 \dots T-1$ and history $h_t = (i_\tau, j_\tau)_{\tau=1}^{t-1}$, specifies $x_t^\theta(h_t) \in \Delta_n$. His action $i_t \sim x_t^\theta(h_t)$ conditional on θ and h_t
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The payoff:

$$\frac{1}{T} \cdot \mathbb{E}_{\theta,h_T}\left[\sum_{t=0}^{T-1} A^\theta_{i_t,j_t}\right]$$

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Question: Why $\min \max = \max \min$?

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Pure strategies are deterministic behavioral strategies (for all possible histories and states). For each pair of pure strategies x, y compute the payoff $A'_{x,y}$. By the construction $V_T(p) = \mathrm{val}[A']$.

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We used **Kuhn's theorem:** for any mixed strategy there is a behavioral strategy with the same payoff and vice-versa.

Example

T-stage RGII with payoffs

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Question: What should P1 do?

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  P2 guesses the state after the first round $\Rightarrow$ the payoff is $\frac{V(p)}{T} \to 0$ as $T \to \infty$.
- **Better Idea:** play the optimal strategy from $A^{\mathrm{NR}}(p) = (1-p) \cdot A^0 + p \cdot A^1$.
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  Guarantees $u(p)$ at every stage, so $V_T(p) \geq u(p)$.

Question: Can P1 do better?

Example

T-stage RGII with payoffs

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Question: What should P1 do?

- **Bad Idea:** play the optimal strategy from the static game $G(p) \equiv G_1(p)$: Top if $\theta = 0$ and Bottom if $\theta = 1$.
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Question: Can P1 do better?

Answer: Not much.

The $\mathrm{Cav}\,[u]$-theorem

Theorem (R.Aumann, M.Maschler, 1960ies)

$$\mathrm{Cav}\,[u](p) \leq V_T(p) \leq \mathrm{Cav}\,[u](p) + \frac{2\|A\|}{\sqrt{T}},$$

where $\|A\| = \max_{i,j,\theta} \left|A^{\theta}_{i,j}\right|$.

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- **Lower bound:** $G_T(p) \leftrightarrow$ a static game with incomplete information $\Rightarrow V_T(p)$ is concave. $V_T(p) \geq u(p) \Rightarrow V_T(p) \geq \mathrm{Cav}\,[u](p)$.
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- **Upper bound:** . . . .

Method 1: the upper bound via Blackwell's approachability

Remark: this method gives a weaker result:

$$\limsup_{T\to\infty} V_T(p) \leq \mathrm{Cav}\,[u](p).$$

No control on the speed of convergence.

Reminder: Blackwell's approachability

Consider a game $\vec{G}_T$ with vector payoff $\vec{A} = \begin{pmatrix} A^0 \\ A^1 \end{pmatrix}$.

Definition: A set $C \subset \mathbb{R}^2$ is approachable by P2 $\Leftrightarrow$ P2 has a behavioral strategy such that the average payoff approaches C in the limit, no matter what P1 is doing:

$$\mathbb{E}\left(\text{dist}\left(\frac{1}{T}\sum_{t=0}^{T-1}\vec{A}_{i_t,j_t}, \quad C\right)\right) \to 0 \quad \text{as } T \to \infty.$$

Theorem (Blackwell)

$L(\alpha) = (-\infty, \alpha_0] \times (-\infty, \alpha_1]$ is approachable by P2 if

$$\text{val}[(1-q)A^0 + qA^1] \leq (1-q)\alpha_0 + q \cdot \alpha_1 \quad \text{for any } q \in [0,1].$$

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Remark: $\operatorname{val}[(1-q)A^0 + qA^1] = u(q)$

Application to RGII: the upper bound on $V_T(p)$

Picking alphas: $l(q) = (1 - q) \cdot \alpha_0 + q \cdot \alpha_1$ is the tangent line to the graph of $\mathrm{Cav}\,[u]$ at p:

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Method 2: the upper bound via martingales of posterior beliefs

Remark: this method allows to control the error term

$$V_T(p) \leq \mathrm{Cav}\,[u](p) + \frac{2\|A\|}{\sqrt{T}}$$

Variation of posterior beliefs

Fix some strategy x of Player 1.

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- This is a payoff in a game $A^{\mathrm{NR}}(p_{t+1})$ if P2 plays his optimal strategy from $A^{\mathrm{NR}}(p_t)$. Since $\left|A^{\mathrm{NR}}_{i,j}(p_{t+1}) - A^{\mathrm{NR}}_{i,j}(p_t)\right| \leq 2\|A\| \cdot |p_{t+1} - p_t|$

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- By Jensen's inequality and the martingale property $\mathbb{E}[u(p_t)] \leq \mathbb{E}[\mathrm{Cav}\,[u](p_t)] \leq \mathrm{Cav}\,[u](\mathbb{E}p_t) = \mathrm{Cav}\,[u](p)$. □

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For any martingale $\xi_0, \xi_1, \ldots$ on filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$

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Note that $\mathbb{E}[\xi_{t+1} \cdot \xi_t] = \mathbb{E}\big[\mathbb{E}[\xi_{t+1} \cdot \xi_t \mid \mathcal{F}_t]\big] = \mathbb{E}[\xi_t^2]$.

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Proof: Cauchy-Shwartz inequality

$$\mathbb{E}\Big[\sum^{T-1} |\xi_{t+1} - \xi_t|\Big] = \mathbb{E}\Big[\sum^{T-1} 1 \cdot |\xi_{t+1} - \xi_t|\Big] \le \sqrt{\mathbb{E}\Big[\sum^{T-1} 1\Big]} \sqrt{\mathbb{E}\Big[\sum^{T-1} (\xi_{t+1} - \xi_t)^2\Big]}. \quad \square$$

Extensions & references

Extensions:

- Non-binary set of states $\Theta \Rightarrow$ no complications: $\Delta(\Theta)$ replaces $[0,1]$. Continuous Θ and sets of actions are doable (Gensbittel 2015)
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Main references:

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  A good survey of classic results with proofs.
- **M.Maschler, E.Solan, S.Zamir** "Game Theory" 2013
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